Dynamic Quantitative Spatial Models: Caliendo, Dvorkin, and Parro (2019) Kleinman, Liu, and Redding (2023)

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Caliendo, Dvorkin, and Parro (2019)

Equilibrium

- Time-varying fundamentals: $\Theta_t = \{A_t, \kappa_t\}$
- Constant fundamentals: $\bar{\Theta} = \{\Upsilon, H, b\}$
- Parameters: $\{\gamma, \xi, \iota, \alpha, \beta, \theta, \nu\}$
- Static variables: $\{w_t, \pi_t, X_t\}$
- Dynamic variables: $\{L_t, \mu_t, V_t\}$
- Temporary equilibrium: $\{w_t, \pi_t, X_t\}$ that solve the static subproblem given $\{L_t, \Theta_t, \bar{\Theta}\}$
  - Equations: unit price, price index, trade share, market clearing (goods, labor, structures)
- Sequential competitive equilibrium: $\{L_t, \mu_t, V_t, w_t(L_t, \Theta_t, \bar{\Theta})\}$ that solve the dynamic + static subproblems given $\{L_0, \Theta_t, \bar{\Theta}\}$
  - Equations: worker value function, migration share, law of motion for labor
  - Stationary equilibrium: $\{L_t, \mu_t, V_t, w_t(L_t, \Theta_t, \bar{\Theta})\}$ are constant

Data and Estimation

- Data: $\pi_t, w_t L_t + r_t H_t, L_t, \mu_t$
- Calibrate $\gamma, \xi, \alpha, \iota$ from data
- $\beta = 0.99$
- θ from Caliendo and Parro (2015)
- Estimate ν from model-implied migration shares

$$\log\left(\mu_t^{nj,nk}/\mu_t^{nj,nj}\right) = \tilde{C} + \frac{\beta}{\nu}\log\left(w_{t+1}^{nk}/w_{t+1}^{nj}\right) + \beta\log\left(\mu_{t+1}^{nj,nk}/\mu_{t+1}^{nj,nj}\right) + \varpi_{t+1}$$

- Need data for at least two years to estimate ν; otherwise, only need data for one year

Dynamic Hat Algebra

- $\hat{x} = \frac{x'}{x}$
- Some common operations:
  1. $x = \prod_{i=1}^{N} x_i^{\alpha_i} \implies \hat{x} = \prod_{i=1}^{N} \hat{x}_i^{\alpha_i}$
  2. $x = \left(\sum_{i=1}^{N} \alpha_i x_i\right)^{\beta} \implies \hat{x} = \left(\frac{\alpha_i x_i}{\sum_{j=1}^{N} \alpha_j x_j} \hat{x}_i\right)^{\beta}$
- Dynamic hat algebra: change is defined over time $\dot{x}_t = \frac{x_t}{x_{t-1}}$
- Idea: find the equilibrium without knowing fundamentals and solving the DP problem
- Dynamic hat algebra for counterfactuals: difference-in-difference, over time and over different paths of fundamentals

Sequential Competitive Equilibrium (1/2)

- One layer of difference (over time)
- Initialize algorithm with:
  1. Initial allocation: L_0, π_0, X_0, μ_{-1}
  2. A convergent path of changes in fundamentals: $\{\dot{\Theta}_t\}$
- Fixed point algorithm

1. Guess a convergent path for value functions $\{\dot{u}_{t+1}\}$, $\lim_{t\to\infty} \dot{u}_{t+1} = 1$
2. Simulate forward using $\{\dot{u}_{t+1}\}$ to compute $\{\mu_{t+1}\}$:

$$\mu_{t+1}^{nj,ik} = \frac{\mu_t^{nj,ik} \left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu}}{\sum_{m=1}^{N} \sum_{h=0}^{J} \mu_t^{nj,mh} \left(\dot{u}_{t+2}^{mh}\right)^{\beta/\nu}}$$

3. Simulate forward using $\{\mu_{t+1}\}$ to compute $\{L_{t+1}\}$:

$$L_{t+1}^{nj} = \sum_{i=1}^{N} \sum_{k=0}^{J} \mu_t^{ik,nj} L_t^{ik}$$

Sequential Competitive Equilibrium (2/2)

4. Solve the temporary equilibrium for each t to get $\left\{\dot{w}_{t+1}, \dot{P}_{t+1}\right\}$ (see below)
5. Update $\{\dot{u}_{t+1}\}$ by simulating backwards:

$$\dot{u}_{t+1}^{nj} = \frac{\dot{w}_{t+1}^{nj}}{\dot{P}_{t+1}^{nj}} \left(\sum_{i=1}^{N} \sum_{k=0}^{J} \mu_t^{nj,ik} \left(\dot{u}_{t+2}^{ik}\right)^{\beta/\nu} \right)^{\nu}$$

6. Iterate steps 1-5 until convergence

Temporary Equilibrium (1/2)

- Given $\{L_t, w_t\}$ (levels) and $\{\dot{L}_{t+1}, \dot{\Theta}_{t+1}\}$ (changes), solve for $\{\dot{w}_{t+1}\}$
- Another fixed point algorithm

1. For every t, guess $\{\dot{w}_{t+1}\}$
2. Solve for $\{\dot{x}_{t+1}, \dot{P}_{t+1}\}$ from the following non-linear system using fixed point algorithm:

$$\dot{x}_{t+1}^{nj} = \left(\dot{L}_{t+1}^{nj}\right)^{\gamma^{nj}\xi^n} \left(\dot{w}_{t+1}^{nj}\right)^{\gamma^{nj}} \prod_{k=1}^{J} \left(\dot{P}_{t+1}^{nk}\right)^{\gamma^{nj,nk}}$$

$$\dot{P}_{t+1}^{nj} = \left(\sum_{i=1}^{N} \pi_t^{nj,ij} \left(\dot{x}_{t+1}^{ij} \dot{\kappa}_{t+1}^{nj,ij}\right)^{-\theta^j} \left(\dot{A}_{t+1}^{ij}\right)^{\theta^j \gamma^{ij}}\right)^{-1/\theta^j}$$

3. Compute $\{\pi_{t+1}\}$:

$$\pi_{t+1}^{nj,ij} = \pi_t^{nj,ij} \left(\frac{\dot{x}_{t+1}^{ij} \dot{\kappa}_{t+1}^{nj,ij}}{\dot{P}_{t+1}^{nj}}\right)^{-\theta^j} \left(\dot{A}_{t+1}^{ij}\right)^{\sigma^j \gamma^{ij}}$$

Temporary Equilibrium (2/2)

4. Solve for $\{X_{t+1}\}$ from the following linear system:

$$X_{t+1}^{nj} = \sum_{k=1}^{J} \gamma^{nk,nj} \sum_{i=1}^{N} \pi_{t+1}^{ik,nk} X_{t+1}^{ik} + \alpha^{j} \left(\sum_{k=1}^{J} \dot{w}_{t+1}^{nk} \dot{L}_{t+1}^{nk} w_{t}^{nk} L_{t}^{nk} + \iota^{n} \chi_{t+1} \right)$$

where $\chi_{t+1} = \sum_{i=1}^{N} \sum_{k=1}^{J} \frac{\xi^{i}}{1-\xi^{i}} \dot{w}_{t+1}^{ik} \dot{L}_{t+1}^{ik} w_{t}^{ik} L_{t}^{ik}$

5. Update $\{\dot{w}_{t+1}\}$:

$$\dot{w}_{t+1}^{nj} \dot{L}_{t+1}^{nj} w_{t}^{nj} L_{t}^{nj} = \gamma^{nj} \left(1 - \xi^{n}\right) \sum_{i=1}^{N} \pi_{t+1}^{ij,nj} X_{t+1}^{ij}$$

6. Iterate steps 1-5 until convergence

Counterfactual Equilibrium

- Another layer of difference (over different paths of fundamentals)
- Define $\hat{x}_{t+1} = \frac{\dot{x}'_{t+1}}{\dot{x}_{t+1}}$
- Initialize algorithm with:
  1. Baseline allocation: $\{L_t, \mu_t, \pi_t, X_t\}$ (also know "dots")
  2. A counterfactual convergent path of changes in fundamentals: $\{\hat{\Theta}_t\}$
- Timing: unanticipated shocks at $t = 1$
- Algorithm is similar with slightly modified equation systems (essentially do hat algebra again)

Counterfactual Sequential Competitive Equilibrium (1/2)

1. Guess a convergent path for value functions $\{\hat{u}_{t+1}\}$, $\lim_{t\to\infty} \hat{u}_{t+1} = 1$, $\hat{u}_0 = 1$
2. Simulate forward using $\{\dot{u}_{t+1}\}$ to compute $\{\mu_{t+1}\}$:

$$\mu_t'^{nj,ik} = \begin{cases} \mu_0^{nj,ik} & \text{if } t = 0 \\ \dfrac{\mu_1^{nj,ik} \left(\hat{u}_1^{ik}\right)^{\beta/\nu} \left(\hat{u}_2^{ik}\right)^{\beta/\nu}}{\sum_{m=1}^{N} \sum_{h=0}^{J} \mu_1^{nj,mh} \left(\hat{u}_1^{mh}\right)^{\beta/\nu} \left(\hat{u}_2^{mh}\right)^{\beta/\nu}} & \text{if } t = 1 \\ \dfrac{\mu_{t-1}'^{nj,ik} \dot{\mu}_t^{nj,ik} \left(\hat{u}_{t+1}^{ik}\right)^{\beta/\nu}}{\sum_{m=1}^{N} \sum_{h=0}^{J} \mu_{t-1}'^{nj,mh} \dot{\mu}_t^{nj,mh} \left(\hat{u}_{t+1}^{mh}\right)^{\beta/\nu}} & \text{if } t > 1 \end{cases}$$

3. Simulate forward using $\{\mu'_{t+1}\}$ to compute $\{L'_{t+1}\}$:

$$L_{t+1}'^{nj} = \sum_{i=1}^{N} \sum_{k=0}^{J} \mu_t'^{ik,nj} L_t'^{ik}$$

Counterfactual Sequential Competitive Equilibrium (2/2)

4. Solve the temporary equilibrium for each t to get $\left\{\hat{w}_{t+1}, \hat{P}_{t+1}\right\}$ (see below)
5. Update $\{\hat{u}_{t+1}\}$ by simulating backwards:

$$\hat{u}_t^{nj} = \begin{cases} \left(\frac{\hat{w}_t^{nj}}{\hat{P}_t^n}\right)\left(\sum_{i=1}^{N}\sum_{k=0}^{J}\mu_{t-1}^{\prime nj,ik}\dot{\mu}_t^{nj,ik}\left(\hat{u}_{t+1}^{ik}\right)^{\beta/\nu}\right)^{\nu} & \text{for } t \geq 2 \\ \left(\frac{\hat{w}_1^{nj}}{\hat{P}_1^n}\right)\left(\sum_{i=1}^{N}\sum_{k=0}^{J}\mu_1^{nj,ik}\left(\hat{u}_1^{ik}\right)^{\beta/\nu}\left(\hat{u}_2^{ik}\right)^{\beta/\nu}\right)^{\nu} & \text{for } t = 1 \end{cases}$$

6. Iterate steps 1-5 until convergence

Counterfactual Temporary Equilibrium (1/2)

1. For every t, guess $\{\hat{w}_{t+1}\}$
2. Solve for $\{\hat{x}_{t+1}, \hat{P}_{t+1}\}$ from the following non-linear system using fixed point algorithm:

$$\hat{x}^{nj}_{t+1} = \left(\hat{L}^{nj}_{t+1}\right)^{\gamma^{nj}\xi^{n}} \left(\hat{w}^{nj}_{t+1}\right)^{\gamma^{nj}} \prod_{k=1}^{J} \left(\hat{P}^{nk}_{t+1}\right)^{\gamma^{nj,nk}}$$

$$\hat{P}^{nj}_{t+1} = \left(\sum_{i=1}^{N} \pi'^{nj,ij}_{t} \dot{\pi}^{nj,ij}_{t+1} \left(\hat{x}^{ij}_{t+1}\hat{\kappa}^{nj,ij}_{t+1}\right)^{-\theta^j} \left(\hat{A}^{ij}_{t+1}\right)^{\theta^j \gamma^{ij}}\right)^{-1/\theta^j}$$

3. Compute $\{\pi'_{t+1}\}$:

$$\pi'^{nj,ij}_{t+1} = \pi'^{nj,ij}_{t} \dot{\pi}^{nj,ij}_{t} \left(\frac{\hat{x}^{ij}_{t+1}\hat{\kappa}^{nj,ij}_{t+1}}{\hat{P}^{nj}_{t+1}}\right)^{-\theta^j} \left(\hat{A}^{ij}_{t+1}\right)^{\theta^j \gamma^{ij}}$$

Counterfactual Temporary Equilibrium (2/2)

4. Solve for $\{X'_{t+1}\}$ from the following linear system:

$$X'^{nj}_{t+1} = \sum_{k=1}^{J} \gamma^{nk,nj} \sum_{i=1}^{N} \pi'^{ik,nk}_{t+1} X'^{ik}_{t+1} + \alpha^j \left(\sum_{k=1}^{J} \hat{w}^{nk}_{t+1} \hat{L}^{nk}_{t+1} w'^{nk}_t L'^{nk}_t \dot{w}^{nk}_{t+1} \dot{L}^{nk}_{t+1} + \iota^n \chi'_{t+1} \right)$$

   where $\chi'_{t+1} = \sum_{i=1}^{N} \sum_{k=1}^{J} \frac{\xi^i}{1-\xi^i} \hat{w}^{ik}_{t+1} \hat{L}^{ik}_{t+1} w'^{ik}_t L'^{ik}_t \dot{w}^{ik}_{t+1} \dot{L}^{ik}_{t+1}$

5. Update $\{\hat{w}_{t+1}\}$:

$$\hat{w}^{nk}_{t+1} \hat{L}^{nk}_{t+1} = \frac{\gamma^{nj} \left(1 - \xi^n\right)}{w'^{nk}_t L'^{nk}_t \dot{w}^{nk}_{t+1} \dot{L}^{nk}_{t+1}} \sum_{i=1}^{N} \pi'^{ij,nj}_{t+1} X'^{ij}_{t+1}$$

6. Iterate steps 1-5 until convergence

Kleinman, Liu, and Redding (2023)

Equilibrium (1/2)

- Parameters: $\{\psi, \theta, \beta, \rho, \mu, \delta\}$
- Fundamentals: $\{\underbrace{z_{it}, b_{it}}_{\text{varying}}, \underbrace{\kappa_{nit}, \tau_{nit}}_{\text{constant}}\}$
- Initial condition: $\{\ell_{i0}, k_{i0}\}$
- Endogenous variables: $\{\underbrace{w_{it}, R_{it}}_{\text{static}}, \underbrace{v_{it}, \ell_{it+1}, k_{it+1}}_{\text{dynamic}}\}$

1. Goods market clearing

$$w_{it}\ell_{it} = \sum_{n=1}^{N} S_{nit} w_{nt} \ell_{nt}, \quad S_{nit} = \frac{\left(w_{it}(\ell_{it}/k_{it})^{1-\mu}\tau_{nit}/z_{it}\right)^{-\theta}}{\sum_{m=1}^{N}\left(w_{mt}(\ell_{mt}/k_{mt})^{1-\mu}\tau_{nmt}/z_{mt}\right)^{-\theta}}, \quad T_{int} = \frac{S_{nit} w_{nt} \ell_{nt}}{w_{it}\ell_{it}}$$

2. Capital market clearing

$$R_{it} = \left(1 - \delta + \frac{1-\mu}{\mu}\frac{w_{it}\ell_{it}}{p_{it}k_{it}}\right), \quad p_{nt} = \left[\sum_{i=1}^{N}\left(w_{it}\left(\frac{1-\mu}{\mu}\right)^{1-\mu}(\ell_{it}/k_{it})^{1-\mu}\tau_{nit}/z_{it}\right)^{-\theta}\right]^{-1/\theta}$$

Equilibrium (2/2)

3. Worker value function

$$v_{nt}^{w} = \log b_{nt} + \log\left(\frac{w_{nt}}{p_{nt}}\right) + \rho \log \sum_{g=1}^{N} \left(\exp(\beta \mathbb{E}_t v_{gt+1}^{w} / \kappa_{gnt})\right)^{1/\rho}$$

4. Population flow

$$\ell_{gt+1} = \sum_{i=1}^{N} D_{igt}\ell_{it}, \quad D_{igt} = \frac{\left(\exp(\beta \mathbb{E}_t v_{gt+1}^{w} / \kappa_{git})\right)^{1/\rho}}{\sum_{m=1}^{N} \left(\exp(\beta \mathbb{E}_t v_{mt+1}^{w} / \kappa_{mit})\right)^{1/\rho}}, \quad E_{git} = \frac{\ell_{it} D_{igt}}{\ell_{gt+1}}$$

5. Low of motion for capital

$$k_{it+1} = (1 - \varsigma_{it}) R_{it} k_{it}, \quad \varsigma_{it}^{-1} = 1 + \beta^{\psi} \left(\mathbb{E}_t \left[R_{it+1}^{\frac{\psi-1}{\psi}} \varsigma_{it+1}^{-\frac{1}{\psi}}\right]\right)^{\psi}$$

Existence and Uniqueness of Steady State

- Sufficient condition from Allen, Arkolakis, and Li (2023) generalizing Allen and Arkolakis (2014)

- Coefficient matrix of parameters (exponents) in the system of equation at steady state

$$\mathbf{A} = \begin{bmatrix} -\theta & 0 & 0 & 0 \\ \theta(1-\mu) & 1+\theta\mu & 1 & 0 \\ \beta/\rho & -\beta/\rho & 1 & -\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Existence and uniqueness of equilibrium if spectral radius of $\mathbf{A}$ is less than or equal to one

Data and Estimation

- Data: $\ell_t, k_t, w_t\ell_t + r_tk_t, D_t, S_t$

- Calibration

$$\psi = 1, \quad \theta = 5, \quad \beta = (0.95)^5, \quad \rho = 3\beta, \quad \mu = 0.65, \quad \delta = 1 - (0.95)^5$$

Do comparative statics with respect to each parameter

Dynamic Hat Algebra - Static (1/2)

$$\dot{p}_{it+1} = \left(\sum_{m=1}^{N} S_{imt} \left(\dot{\tau}_{imt} \dot{w}_{mt+1} \left(\dot{\ell}_{mt+1} / \dot{k}_{mt+1} \right)^{1-\mu} / \dot{z}_{mt+1} \right)^{-\theta} \right)^{-1/\theta}$$

$$\dot{S}_{nit+1} = \frac{\left(\dot{\tau}_{nit+1} \dot{w}_{it+1} (\dot{\ell}_{it+1} / \dot{k}_{it+1})^{1-\mu} / \dot{z}_{it+1} \right)^{-\theta}}{\sum_{m=1}^{N} S_{nmt} \left(\dot{\tau}_{nmt+1} \dot{w}_{mt+1} (\dot{\ell}_{mt+1} / \dot{k}_{mt+1})^{1-\mu} / \dot{z}_{mt+1} \right)^{-\theta}}$$

$$\dot{w}_{it+1} \dot{\ell}_{it+1} = \sum_{n=1}^{N} \frac{S_{nit} w_{nt} \ell_{nt}}{\sum_{k=1}^{N} S_{kit} w_{kt} \ell_{kt}} \dot{w}_{nt+1} \dot{\ell}_{nt+1}$$

Apply the algorithm for temporary equilibrium as in CDP

Dynamic Hat Algebra - Dynamic (2/2)

$$\dot{D}_{igt+1} = \frac{\dot{u}_{gt+2}/\dot{\kappa}_{git+1}^{\frac{1}{\rho}}}{\sum_{m=1}^{N} D_{imt}\dot{u}_{mt+2}/\dot{\kappa}_{mit+1}^{\frac{1}{\rho}}}$$

$$\ell_{gt+1} = \sum_{i=1}^{N} D_{igt}\ell_{it}$$

$$k_{it+1} = (1-\varsigma_{it})R_{it}k_{it}$$

$$(R_{it} - (1-\delta)) = \frac{\dot{p}_{it+1}\dot{k}_{it+1}}{\dot{w}_{it+1}\dot{\ell}_{it+1}}(R_{it+1} - (1-\delta))$$

$$\varsigma_{it+1} = \beta^{\psi} R_{it+1}^{\psi-1} \frac{\varsigma_{it}}{1-\varsigma_{it}}$$

$$\dot{u}_{it+1} = \left(\dot{b}_{it+1}\frac{\dot{w}_{it+1}}{\dot{p}_{it+1}}\right)^{\frac{\beta}{\rho}} \left(\sum_{g=1}^{N} D_{igt}\dot{u}_{gt+2}/\dot{\kappa}_{git+1}^{\frac{1}{\rho}}\right)^{\beta}$$

Apply the algorithm for sequential equilibrium as in CDP

Spectral Analysis

- Analytical characterization of the transition path
- Linearized equilibrium conditions

$$\tilde{\mathbf{x}}_{t+1} = \mathbf{P}\tilde{\mathbf{x}}_t + \mathbf{R}\tilde{\mathbf{f}}$$

  - $\tilde{\mathbf{x}}_t = \log \mathbf{x}_t - \log \mathbf{x}^*$
  - $\mathbf{P}, \mathbf{R}$ are functions of parameters and observables
  - Stability: spectral radius of $\mathbf{P}$ is less than one
- Eigendecomposition: $\mathbf{P} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}$
  - Eigenvalue λ_k and eigenvector $\mathbf{u}_k$
- Eigen-shock: $\tilde{\mathbf{f}}_k = \mathbf{R}^{-1}\mathbf{u}_k$, with speed of convergence determined by λ_k
  - Higher λ_k, higher persistence, slower convergence
  - Rank shocks by persistence
- All shocks as linear combinations of eigen-shocks; coefficients obtained from linear projection
  - Loadings: importance of each eigen-component and eigenvalue