Endogenous Production Networks with Fixed Costs*

Emmanuel Dhyne[a] Ayumu Ken Kikkawa[b,d] Xianglong Kong[c]
Magne Mogstad[c,d] Felix Tintelnot[c,d]

[a]National Bank of Belgium
[b]Sauder School of Business, UBC
[c]University of Chicago
[d]NBER

October 12, 2023

Abstract

We develop a model of endogenous production networks with fixed costs in the formation of links between firms. We show that the closed economy equilibrium is unique if the set of feasible networks consists only of networks that are acyclic and the buyer initiates the link formation while having full bargaining power in price negotiations with the supplier. We provide examples of multiple equilibria if the supplier initiates the link formation in both cyclic and acyclic feasible networks or if the buyer initiates the link formation in a cyclic production network. We take the acyclic production network model to Belgian data on firm-to-firm production networks and show that it matches well the salient features of the network. The model generates substantial churn in domestic firm-to-firm linkages in response to trade shocks, while delivering only moderately different welfare changes compared to a model with fixed linkages.

*We are grateful to two anonymous referees, the editor (Cecile Gaubert), Costas Arkolakis, David Baqaee, Magnus Tolum Buus, Jonathan Eaton, Emmanuel Farhi, Teresa Fort, Matthew Grant, Basile Grassi, Keith Head, Oleg Itskhoki, Rob Johnson, Harry Li, Alejandra Lopez Espino, Ferdinando Monte, Yuhei Miyauchi, Andrei Nagornyi, Ezra Oberfield, and Alireza Tahbaz-Salehi for useful comments and suggestions. The views expressed in this paper are those of the authors and do not necessarily reflect the views of the National Bank of Belgium or any other institution with which one of the authors is affiliated. The model presented in this paper is a substantially revised version of the one circulated in the NBER working paper #25120 entitled "Trade and Domestic Production Networks."

1 Introduction

Firms interact in a network of buyer-supplier relationships in the production of goods and services. Shocks to the price of supplier inputs affect the costs at which a firm can produce, and shocks to buyer demand affect how much a firm can sell. But what determines the shape of the firm-to-firm production network in the first place, and what are the aggregate implications of the reshuffling of firm-to-firm linkages in response to shocks?

It is well known that the firm-to-firm production network is sparse—only a few of the feasible connections in the network are chosen—and that larger firms have more connections with other firms. To rationalize these patterns, it is intuitive to think that the formation of links in the production networks is costly and features a cost that is easier for larger firms to overcome. As in the seminal works by Melitz (2003) and Antràs and Helpman (2004) that characterize the extensive margin of firm-level international trade, fixed costs associated with establishing a buyer-supplier relationship may play an important role in governing the endogenous formation of the domestic production network. Yet, to date only a few attempts have been made to model the formation of the domestic production network under fixed costs, and little is known about key model properties such as equilibrium uniqueness.[1]

In this paper, we develop a model in which firms combine inputs produced by other firms and labor to produce differentiated products that can be sold to either households or other firms. The production and utility functions can flexibly accommodate heterogeneity in total factor productivity, household taste for particular producers, and firm-pair-specific input efficiency. We assume that both the production and utility functions are of the constant elasticity of substitution type (CES, Dixit and Stiglitz, 1977), sharing a common elasticity of substitution parameter.[2] Establishing a buyer-supplier relationship requires the payment of a fixed cost, paid in units of labor.

In the theory section, we consider two types of production networks. The first network is completely flexible and allows for cycles in the production network, which means that one firm can sell to another firm, and the buyer firm can directly or indirectly, via sales to other firms, also sell to the supplier. The second network is acyclic, which rules out a cyclical relationship between two firms. In other words, in an acyclic production network, the lower triangular matrix in the firm-to-firm sales matrix is full of zeros, with non-zero elements allowed only in the upper triangular part of the firm-to-firm sales matrix. Furthermore, we consider two forms of network formation: buyer- versus supplier-initiated link formation. Whoever pays the fixed cost for building the relationship must receive additional variable profits that warrant the expense. When the network is supplier-initiated, then, in line with other recent research, we assume that the supplier charges a monopolistically competitive markup to both firms and households.[3] When the buyer

[1] See Bernard and Moxnes (2018) and Carvalho and Tahbaz-Salehi (2019) for two recent surveys of the production networks literature.

[2] The assumption that the CES parameter is the same in both the production and utility functions is ubiquitous in recent work on production networks. See, for example, Lim (2018), Huneeus (2018), Dhyne, Kikkawa, Komatsu, Mogstad, and Tintelnot (2022a), Bernard, Dhyne, Magerman, Manova, and Moxnes (2022), and Arkolakis, Huneeus, and Miyauchi (2023).

[3] See Lim (2018) and Bernard et al. (2022). Different approaches are taken in Dhyne, Kikkawa, and Magerman

initiates the link formation and pays the fixed cost, we assume she charges a monopolistically competitive markup to the household but has full bargaining power with the supplier such that the supplier sells to her at marginal cost. While restrictive, we make this latter assumption as it will play a key role in the tractability of the buyer-initiated network formation.

When the formation of the network is supplier-initiated, the firm's problem of who to sell to is additively separable across its potential buyers since the firm has a constant marginal cost. Perhaps not so obvious is that the decision problem of a buying firm in the formation of a buyer-initiated network is similarly additively separable across its suppliers. In general, as analyzed in Antràs, Fort, and Tintelnot (2017), the problem of a buying firm selecting its suppliers at the expense of a fixed cost leads to an interdependent decision problem, as adding one supplier affects the marginal benefit of adding another supplier. However, since here the elasticity of substitution in the production function and the demand elasticity are identical, the substitution and scale effects associated with adding a supplier exactly cancel out each other, making the decision problem of adding a supplier as simple as the decision problem of adding a buyer. We find that if the technology is such that all feasible production networks are acyclic—and the buyer firm pays the fixed cost and initiates the network formation—it is a) straightforward to solve for the equilibrium and b) the closed economy equilibrium is unique. If, however, the technology is such that cycles can occur in the production network, we find that multiple equilibria are possible regardless of whether the buyer or the supplier pays the fixed cost of link formation. Furthermore, even in an acyclic production network, both the tractability and the guaranteed uniqueness of the closed economy equilibrium vanish if the supplier initiates the network formation.

These findings raise the question of the degree to which domestic production networks in the data can be approximated by an acyclic network. To answer this question, we turn to data on firm-to-firm transactions based on the value-added tax (VAT) registry for Belgium (see Dhyne, Magerman, and Rubínová, 2015). We find that around 80 percent of the transactions are consistent with an acyclic network structure and that dropping those transactions that are inconsistent with an acyclic network structure still provides a high correlation with the coefficients from the sectoral input-output matrix obtained from the unrestricted data. While the assumption of an acyclic network is restrictive, our findings suggest that such an acyclic network can approximate the full data on firm-to-firm transactions in several important dimensions.

We then parameterize our quantitative endogenous network model using moments from the Belgian data, focusing on the acyclic network formation model with the links initiated by the buyer. Since Belgium is a small open economy, import and export transactions play an important role in firm activities. We therefore extend the acyclic endogenous network formation model to incorporate imports and exports. Although our closed economy uniqueness result does not apply to the small open economy model, we propose a method to visually inspect the uniqueness of the equilibrium from numerical simulations. We find that the small open economy equilibrium indeed appears to be unique at the estimated parameters. Furthermore, the parameterized model fits

(2022b) and Alviarez, Fioretti, Kikkawa, and Morlacco (2022), but we do not consider their pricing formula here as doing so would considerably complicate the network formation.

well the targeted statistics about the Belgian firm-to-firm production network and the import and export intensity and participation rate.

To study the aggregate implications of the endogenous reshuffling of firm-to-firm linkages in response to shocks, we explore the predictions of the parameterized model on the effects of an aggregate decline in the costs of international trade. We contrast the results from an endogenous production network with those from a production network with the same initial linkages between firms but in which the extensive margin of firm-to-firm linkages is held fixed in response to the shock. Our quantitative results show two main patterns. First, the importance of the endogenous adjustment in the extensive margin of firm-to-firm linkages is asymmetric between declines and increases in the costs of international trade. For declines in the costs of trade, we find a larger adjustments in the domestic firm-to-firm linkages than for increases in the costs of trade. In line with this asymmetric response of the domestic firm-to-firm production network, the differences in the changes of real incomes between the fixed and endogenous network models are also asymmetric. Second, for plausible magnitudes of the changes in the costs of foreign trade, the welfare effects from endogenous linkages in the firm-to-firm production network are moderate despite the presence of a significant churn in both the domestic firm-to-firm linkages and firm-level export/import participation. This finding implies that the response of the extensive margin of firm-to-firm linkages to shocks alone is not sufficient to indicate that accounting for the extensive margin adjustment is quantitatively relevant for welfare.

Our paper contributes to the literature on endogenous production networks. In this literature, researchers have broadly taken three different approaches to the formation of extensive margin network linkages. The first approach considers the formation of a firm-to-firm link requiring the payment of a fixed cost per link. Most closely related to this approach are the works by Lim (2018), Huneeus (2018), and Bernard et al. (2022). In those papers, the selling firm initiates the link formation and charges a constant markup to its buyer firms.[4] A second approach to production network formation has been one of buyer and supplier search. Miyauchi (2021) develops a model of undirected supplier search that features complementarities in the search decision. Arkolakis et al. (2023) establish conditions for uniqueness in a model in which firms make search decisions for suppliers and buyers in a model of regions. Consistent with our quantitative results, their single sector model implies the absence of a first-order welfare effect from network endogeneity as long as the overhead costs of marketing are paid in units of labor (which is the specification for the units of overhead costs for which Dhyne et al., 2022a provide empirical support). Demir, Fieler, Xu, and Yang (2023) model the directed search for suppliers combined with an endogenous quality choice of firms, which can lead to assortative matching of firms in terms of quality. A third

[4] Yet another approach to endogenous network formation with a fixed cost is to take the connections between firms as exogenous but model the entry/exit decision of firms who incur a fixed cost to produce. Taschereau-Dumouchel (2020) solves the social planner's problem for such an economy. Acemoglu and Tahbaz-Salehi (2020) show the non-uniqueness of an endogenous production network equilibrium when firms pay a fixed cost to enter and negotiate over prices in firm-to-firm transactions. Despite the non-uniqueness of equilibrium, they focus on the "greatest full equilibrium," the equilibrium that has the largest set of active firms, and conduct comparative statics for this particular equilibrium.

approach to production network formation has been to assume that the network formation arises from individual firms deciding between a set of technologies. Such an approach is taken in the works by Oberfield (2018), Boehm and Oberfield (2020), Acemoglu and Azar (2020), Eaton, Kortum, and Kramarz (2022), and Panigrahi (2021).[5] Our contribution to the literature on endogenous production networks is to illustrate the potential for multiplicity of equilibria in network formation models with fixed costs and to derive a set of model restrictions that are sufficient to avoid this multiplicity of equilibria in a closed economy.

Our paper also relates to the literature on production networks that takes as given the extensive margin of firm-to-firm linkages. Building on the seminal work by Hulten (1978), several works aim to derive the importance of networks for understanding aggregate outcomes arising from nonlinearities (e.g., Baqaee and Farhi, 2019a) or distortions (e.g., Baqaee and Farhi, 2019b). Baqaee and Farhi (2021) study the transmission of trade and technology shocks in a multi-country and multi-industry general equilibrium model without fixed costs. Huneeus, Kroft, and Lim (2021) and Dhyne et al. (2022a) take the extensive margin of the domestic production network as given when studying worker-level outcomes in response to trade shocks. Our quantitative findings imply that a tractable and well-behaved endogenous network formation model can lead to aggregate real income changes that are very similar to the ones obtained from models that take the extensive margin of firm-to-firm linkages as exogenous.

The rest of the paper is structured as follows. We present a model of an economy with a fixed extensive margin of firm-to-firm linkages in Section 2. Building on the model with fixed linkages, we then present a model that incorporates the endogenous formation of firm-to-firm linkages in Section 3. In Section 4 we introduce the data, quantify the model, and conduct counterfactual analyses. Section 5 concludes.

2 Economy with Fixed Production Networks

In this section, we construct a model of an economy with fixed firm-to-firm production networks and discuss the existence and uniqueness of the equilibrium. There exists a finite set of firm types, Ω, with each firm type producing a differentiated product. For the remainder of this section, one can interpret each firm type as representing a single firm, leading to a finite number of firms. To guarantee the existence of an equilibrium under an endogenous network in Section 3, however, we will appeal to the view that there are a continuum of firms of each type. Under fixed production networks, firms take buyer-supplier links as given and only decide how much to buy from each of their suppliers and how much labor to hire. We first focus on a closed economy, then extend the model to a small open economy.

Before describing the model, we briefly discuss the notation. Our model is of a production network in the economy, and it describes the trade flows between firms extensively. Whenever

[5] Yet another related approach is to model the process of firms connecting with others as exogenous, as in the pioneering works by Chaney (2014) and Atalay, Hortacsu, Roberts, and Syverson (2011). This exogenous meeting process is similar to the one taken by Jackson and Rogers (2007) when modeling the formation of dynamic social networks, where individual-to-individual networks are formed.

a variable has two subscripts, the first subscript denotes the origin of the good and the second subscript denotes the destination of the good.

2.1 Preferences and Demand

Each consumer supplies one unit of labor inelastically. Consumers are assumed to have identical, homothetic CES preferences over differentiated consumption goods:

$$U = \left(\sum_{k \in \Omega} (\beta_{kH} q_{kH})^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where Ω denotes the set of available products in the economy, k denotes a firm producing the differenciated product, and H denotes domestic final demand from households. The term β_{kH} captures how salient the good produced by firm k is in the final consumption bundle. Since all consumers have the same homothetic CES preferences for consumption, we can write the aggregate final consumer demand (in quantities) for product k, given price p_{kH}, as

$$q_{kH} = \beta_{kH}^{\sigma-1} \frac{p_{kH}^{-\sigma}}{P^{1-\sigma}} E, \tag{2}$$

where E denotes the aggregate expenditure, and P denotes the domestic consumer price index:

$$P = \left(\sum_{k \in \Omega} \beta_{kH}^{\sigma-1} p_{kH}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}. \tag{3}$$

We assume that final goods are substitutes, and therefore $\sigma > 1$.

2.2 Market Structure and Production

Each firm produces a single differentiated product, for which we use i, j, k for indexing. A firm sells its own variety to final consumers and to other firms as an intermediate input, though not all firms sell to other firms, and not all pairs of firms have a buyer-supplier relationship.

Each firm takes the marginal costs of its suppliers, the wage level, aggregate expenditure, and the consumer price index as given. Quantities of purchases are chosen by the buyer of the transaction, taking everyone else's purchase quantities as given.

We assume that a firm sets a price that is equal to an exogenous markup times its marginal cost.[6] Firms set prices that may differ according to the identity of the buyer. When selling to other firms, we assume that supplier j may charge a pair-specific markup when selling to firm k, μ_{jk}. When selling to households, supplier j charges a markup over marginal costs, μ_{jH}.

Firms use a CES input bundle of workers and intermediate inputs with an elasticity of substitution $\sigma > 1$ in the production function. Notice that the elasticity of substitution is the same

[6]Special cases would be that firms charge either a monopolistically competitive markup or no markup. Our formulation is more general at this point and allows for arbitrary pair-specific fixed markups.

for final consumption and production. Given the CES production function, firm j solves a cost minimization problem and chooses its optimal set of inputs, $\{q_{kj}\}$ and ℓ_j, by taking as given input prices $\{p_{kj}\}$ and wage w. We can write the implied unit cost function of firm j as

$$c_j = \frac{1}{\phi_j}\left(\sum_{k\in Z_j}\alpha_{kj}^{\sigma-1}p_{kj}^{1-\sigma}+\alpha_{Lj}^{\sigma-1}w^{1-\sigma}\right)^{\frac{1}{1-\sigma}}. \tag{4}$$

The term Z_j denotes the set of suppliers from which firm j is eligible to purchase inputs. Given the fixed network structure, firm j takes Z_j as given. The first term in the cost function, ϕ_j, denotes the exogenous total factor productivity of firm j. Inside the parentheses, the term α_{kj} reflects how salient the good produced by supplier k is as an input for firm j, and p_{kj} denotes the price that supplier k charges for its input to firm j. Furthermore, the term $\alpha_{Lj} > 0$ captures the importance of labor input for firm j. The strict inequality ensures that all firms use a positive quantity of labor in equilibrium.

2.3 Firm Cost Shares, Sales, and Profits

The share of variable costs by firm j that is spent on intermediate inputs produced by firm $k \in Z_j$ is

$$s_{kj} = \frac{p_{kj}q_{kj}}{c_j q_j} = \frac{\alpha_{kj}^{\sigma-1}p_{kj}^{1-\sigma}}{\Theta_j}, \tag{5}$$

where q_{kj} is the sales quantity from firm k to firm j, and q_j is the total sales quantity of firm j. The term Θ_j, defined below, denotes the *sourcing capability* of firm j.

$$\Theta_j = \sum_{k\in Z_j}\alpha_{kj}^{\sigma-1}p_{kj}^{1-\sigma}+\alpha_{Lj}^{\sigma-1}w^{1-\sigma}, \tag{6}$$

where Z_j denotes the *sourcing strategy* of firm j, following the terminology of Antràs et al. (2017).[7] Firm j spends a larger fraction of variable costs on inputs produced by firm k if the saliency term α_{kj} is large or the price p_{kj} is low, relative to its sourcing capability. Analogously, the share of variable costs by firm j that is spent on labor is

$$s_{Lj} = \frac{w\ell_j}{c_j q_j} = \frac{\alpha_{Lj}^{\sigma-1}w^{1-\sigma}}{\Theta_j}. \tag{7}$$

Firms' total sales consist of the sum of sales to final consumers and sales to other firms. Let firm j's total sales be

$$x_j = \underbrace{\beta_{jH}^{\sigma-1}\mu_{jH}^{1-\sigma}\phi_j^{\sigma-1}\Theta_j\frac{E}{P^{1-\sigma}}}_{\text{Sales to final consumers}} + \sum_{k\in B_j}\underbrace{\alpha_{jk}^{\sigma-1}\mu_{jk}^{1-\sigma}\phi_j^{\sigma-1}\Theta_j\frac{x_k}{\Theta_k\bar{\mu}_k}}_{\text{Sales to firm } k = p_{jk}q_{jk}}, \tag{8}$$

[7] The larger sourcing capability Θ_j, the lower is the marginal cost of the firm, as $c_j = \frac{1}{\phi_j}\Theta_j^{\frac{1}{1-\sigma}}$.

where B_j denotes the *selling strategy* of firm j, or, in other words, the set of buyers to which firm j sells, and $\bar{\mu}_j$ denotes the quantity-weighted average markup of firm j, which is defined as

$$\bar{\mu}_j = \sum_{k \in B_j \cup \{H\}} \frac{q_{jk}}{q_j} \mu_{jk}. \tag{9}$$

Recall that the firm charges an arbitrary markup to final consumers and a pair-specific markup to other firms. Hence, $\bar{\mu}_j$ depends on the distribution of firm j's sales. We can write the *variable* profits of firm j as

$$\pi_j^{var} = \left(1 - \frac{1}{\bar{\mu}_j}\right) x_j. \tag{10}$$

Note that the sales and profits of a firm depend on its connected suppliers and buyers, which we endogenize in Section 3. All links in the economy can be summarized by a matrix, with element (j, k) being one if firm j sells to firm k and zero otherwise. Then for a single firm j, sourcing strategy Z_j is the j-th column of the matrix, and selling strategy B_j is the j-th row. Indeed, the network structure can be summarized by only the columns or only the rows of the matrix. We use both column and row to describe the links of a firm because it allows us to distinguish the links the firm chooses from the links the firm takes as given. For example, if only buyers can form links—as in one of the cases we consider in Section 3—then from firm j's perspective, Z_j is a choice and B_j is given. In this case, we can represent B_j using $\{Z_k\}_{k\in\Omega}$ in equilibrium.

2.4 Aggregation and Fixed Network Equilibrium

We now describe the aggregation of our model, discuss how firm profits are redistributed to consumers, and define the equilibrium. In the model with a fixed production network, we abstract from fixed costs of linkage formation, and hence $\pi_j = \pi_j^{var}$.

We assume that the set of firms is fixed and that firm profits are distributed to workers. Hence, aggregate household expenditure is given by

$$E = wL + \sum_{j\in\Omega} \pi_j. \tag{11}$$

Labor market clearing implies that labor income is equal to firms' labor costs:

$$wL = \sum_{j\in\Omega} \frac{1}{\bar{\mu}_j} s_{Lj} x_j. \tag{12}$$

The fixed network equilibrium is defined as follows.

Definition 1. *Given*

a network structure, $\{Z_j\}_{j\in\Omega}$ *or* $\{B_j\}_{j\in\Omega}$,
a set of markups for sales to households, $\{\mu_{jH}\}_{j\in\Omega}$,
a set of markups for sales to firms, $\{\mu_{jk}\}_{j,k\in\Omega}$,

an aggregate labor supply, L, and

a normalization of the wage level, w,

a ***fixed network equilibrium*** *is characterized by*

firm-level costs, $\{c_j\}_{j\in\Omega}$,

firm-level labor demand, $\{\ell_j\}_{j\in\Omega}$,

quantities of goods purchased by households, $\{q_{jH}\}_{j\in\Omega}$,

quantities of goods purchased by firms, $\{q_{jk}\}_{j,k\in\Omega}$,

a price index for the consumer, P, and

an aggregate expenditure, E,

such that equations (2), (3), (4), (5), (7), (8), (9), (10), (11), *and* (12) *hold.*

2.5 Existence and Uniqueness of Fixed Network Equilibrium

We next discuss sufficient conditions for the existence and uniqueness of the fixed network equilibrium, with details in Appendix A.1. We start by pointing out that equation (4) is a linear system that characterizes $c_j^{1-\sigma}$. Therefore, if the coefficient matrix is invertible given the network and markups, there exists a unique vector of costs that solve the linear system. Furthermore, if the spectral radius of the coefficient matrix—i.e., the maximum of the absolute values of its eigenvalues—is less than one, the cost vector is guaranteed to be positive. In this case, the sourcing capabilities of all firms and the price index P are uniquely determined.

We provide two sets of sufficient conditions that ensure that there exists a unique vector of positive costs. The first is when the matrix is strictly diagonally dominant.[8] The second is when the production network is acyclic. When the production network is acyclic, the coefficient matrix is lower triangular. Either one of these two conditions ensures that there is a unique vector of positive costs.

In addition to the uniqueness of firm-level costs, if the vector of firm-level sales and profits are unique, then the equilibrium aggregate expenditure E is also unique. Less clear, however, is that there exists a unique sales vector because equation (8) is a system of nonlinear equations in x_j as a result of the endogenous average markup $\bar{\mu}_j$. In general, a system of non-linear equations may have multiple solutions. We provide two alternative assumptions on the markups, which ensure a unique sales vector.

First, suppose that firms do not charge markups in firm-to-firm transactions (i.e., $\mu_{jk} = 1$ for all $k \in B_j$ and $j \in \Omega$). In this case, firms do not generate profits from their sales to other firms. Hence, equation (11) simplifies to

$$E = wL + \sum_{j\in\Omega} \pi_{jH} = wL + \sum_{j\in\Omega} \left(1 - \frac{1}{\mu_{jH}}\right) \beta_{jH}^{\sigma-1} \mu_{jH}^{1-\sigma} \phi_j^{\sigma-1} \Theta_j \frac{E}{P^{1-\sigma}}, \tag{13}$$

[8] This is when $\sum_{k\neq j} \mathbb{I}\{k \in Z_j\} \left(\frac{\phi_j \alpha_{kj}}{\mu_{kj}}\right)^{\sigma-1} < 1$ for all $j \in \Omega$. The condition imposes restrictions on the magnitudes of own productivity, ϕ_j, and link specific productivity, α_{kj}, given the network structure and markups. See Appendix A.1 for details.

where π_{jH} is firm j's profits from selling to final consumers. The above equation implies that the aggregate expenditure, E, can be solved explicitly.

Second, suppose that each firm charges a markup that does not vary across the identity of its buyers and is no less than one (i.e., $\mu_{jH} = \mu_{jk} \geq 1$ for all $k \in B_j$ and $j \in \Omega$). Note that this assumption still allows the markup to vary across suppliers. In this case, equation (8) becomes a system of linear equations since the average markup $\bar{\mu}_j$ no longer depends on firm j's distribution of sales. In Proposition 2 in Appendix A.1, we show that when this restriction on markups and the conditions for the uniqueness of costs are satisfied, the sales vector is also unique, yielding a unique solution for the equilibrium aggregate expenditure, E.

With a unique vector of firm-level sales, x_j, and firm-level cost, c_j, quantities of firms' sales, q_j, can be computed. Firms' quantities of sales to final consumers, q_{jH}, can also be obtained from equation (2) given firm-level marginal costs and aggregate expenditure. Finally, quantities of firm-to-firm sales, q_{jk}, can be computed from equation (5). The uniqueness of the equilibrium follows.

2.6 Extension to a Small Open Economy

The model presented so far can be extended to a small open economy by additionally considering a foreign supplier and a foreign buyer. As with firm-to-firm linkages, we take as given the set of firms that import from the foreign supplier and the set of firms that export to the foreign buyer. We also assume that domestic final consumers do not buy directly from the foreign supplier. In a small open economy, importers take as given the foreign supplier's output price, $\{p_{Fj}\}_{j\in\Omega}$, which enters their cost function in (4) as another input. The foreign buyer is assumed to have a CES preference over domestic goods, and the resulting firm-level exports can be written as $q_{jF} = \beta_{jF}^{\sigma-1}\tau_{jF}^{-\sigma}p_{jF}^{-\sigma}D_F$, where β_{jF} is the firm-specific foreign demand shifter, τ_{jF} is the firm-specific cost of exporting, and D_F is an exogenously given foreign demand shifter.

In Appendix A.5 we provide a full set of equilibrium equations of the small open economy. In a small open economy, the domestic wage, w, is now an additional equilibrium variable that is determined by the aggregate trade balance condition. One can follow the same steps as in Section 2.5 to establish the uniqueness of the other equilibrium variables, given a level of the domestic wage. However, our theoretical results do not extend to showing that the equilibrium wage level in a small open economy is unique.

3 Economy with Endogenous Network Formation

We move on to develop a model of trade with endogenous network formation, allowing buyer-supplier relationships to change in response to shocks. The model builds on the theoretical framework presented in Section 2. We assume the same preferences, demand functions, and production technology as in the model with fixed production networks.

The key extension from the previous section is that firms now optimally choose their set of links. Specifically, we consider two cases of network formation: (1) buyers initiate link formations with suppliers, and (2) suppliers initiate link formations with buyers. Forming linkages is costly, and initiating firms incur a random, firm-pair-specific fixed cost measured as f_{jk} units of labor. The realizations of fixed costs are known to the firm at the time it forms the links. When the network is supplier initiated, we assume that the supplier charges a monopolistically competitive markup to both firms and households. When the buyer initiates the link formation and pays the fixed cost, we assume she charges a monopolistically competitive markup to the household but has full bargaining power with the supplier such that the supplier sells to her at marginal cost. In both cases, we do not allow for side payments.

Furthermore, we consider two distinct configurations of technology parameters, $\{\alpha_{kj}\}$, that imply two different structures of the domestic production network. The first configuration of technology allows any firm's output to be used as input for production by any other firm, which can generate cycles in the production network. In a cyclic network structure, one firm can sell to another firm and the buyer firm can directly or indirectly, via sales to other firms that also sell to the supplier. The second configuration of technology imposes the restriction that only the output of some firms is useful as an input for other firms. Specifically, we assume that the firms are indexed following an ordering such that $\alpha_{kj} = 0$ whenever $k > j$. This configuration of technology generates an acyclic production network, which rules out cycles in the relationship between two firms. In other words, in an acyclic production network, non-zero elements are contained only in the upper triangular part of the firm-to-firm sales matrix.

We start by characterizing the equilibrium of the economy when either the buyers or the suppliers initiate link formations. We next establish the uniqueness of the equilibrium when buyers initiate link formations under an acyclic network structure. We then provide examples of multiple equilibria when buyers initiate link formations under a cyclic network structure and when suppliers initiate link formations under either a cyclic or acyclic network structure. Finally, we extend the model to a small open economy.

Before proceeding, we note that our model of endogenous production networks is distinct from but closely related to other models on endogenous production networks with fixed costs in the literature. Table 1 provides a summary of the common and distinguishing features of these network formation models. The table indicates what modifications one would need to make in the other papers to apply our results from this section to their settings. It shows that in contrast to our model, other models in the literature typically consider a continuum of firms, cyclic networks, and supplier-initiated contacts.

3.1 Equilibrium Characterizations

3.1.1 Equilibrium when buyers initiate link formations

We begin by considering an economy where only buyers can initiate linkages with other firms. Buyers choose the optimal set of suppliers within their potential suppliers and pay fixed costs in terms of labor that vary for each firm-pair. Upon establishing links, firms do not make profits from

Table 1: Comparison of endogenous production network models with fixed costs

	Lim (2018)	Huneeus (2018)	Bernard et al. (2022)	This paper
Decision horizon	Static	Dynamic	Static	Static
Firm types	Continuous	Continuous	Continuous	Finite with a continuum of firms within type
Preferences	CES with firm-specific demand shifter	CES with firm-specific demand shifter	CES with common demand shifter	CES with firm-specific demand shifter
Technology	CES with common technology shifter for intermediates	Nested CES with common technology shifter within nest for intermediates	Cobb-Douglas between labor and CES composite of intermediates with common technology shifter	CES with firm-pair-specific technology shifter for intermediates
Common elasticity of substitution[1]	Yes	No	Yes (across intermediates)	Yes
Network structure	Cyclic	Cyclic	Cyclic	Acyclic or cyclic
Contact initiator	Supplier	Supplier	Supplier	Buyer or supplier

Notes: [1]Common elasticity of substitution refers to the elasticities of substitution being the same within preferences and technology.

sales to other firms as buyers are assumed to have all the bargaining power. Therefore, variable profits are proportional to firm-level sales to final consumers. We assume that firms observe the marginal costs of their potential suppliers and take them as given when solving for the optimal sourcing strategy. In the final demand market, firms engage in monopolistic competition where they charge a common markup of $\mu_H = \frac{\sigma}{\sigma-1}$ while taking as given the household demand shifter, $\frac{E}{P^{1-\sigma}}$, when maximizing profits.

Given its own sourcing strategy, Z_j, the profits of firm j are equal to variable profits less the fixed costs of link formation, $\sum_{k\in Z_j} f_{kj}w$:

$$\pi_j(Z_j) = \left(1 - \frac{1}{\mu_H}\right) \beta_{jH}^{\sigma-1} \mu_H^{1-\sigma} \phi_j^{\sigma-1} \Theta_j(Z_j) \frac{E}{P^{1-\sigma}} - \sum_{k\in Z_j} f_{kj}w. \tag{14}$$

We assume that firm j exogenously meets a set of potential suppliers, $\mathbf{Z}_j$. Firm j then endogenously decides on the set of suppliers:

$$\max_{Z_j} \ \pi_j(Z_j) \quad \text{s.t.} \quad Z_j \subseteq \mathbf{Z}_j. \tag{15}$$

When deciding on whether to add a supplier to its sourcing strategy, the firm is trading off a reduction in its variable cost (leading to higher variable profits) for an increase in its fixed cost. We impose a tie breaking rule that a firm includes an potential supplier in its sourcing strategy when the net benefit is zero. Note that the elasticity of substitution in the production function and the demand elasticity are identical. From equation (14), the marginal benefit and marginal cost of adding a specific supplier are additively separable from those of adding other suppliers. Therefore, a firm's decision problem of adding a supplier is as simple as the decision problem of adding a buyer. In contrast to Antràs et al. (2017), in which the problem of a buying firm selecting its suppliers is an interdependent decision problem, here the substitution and scale effects associated with adding a supplier exactly cancel out each other, and the buying firm is able to choose whether to source from each supplier independent of other suppliers.

The definition of the equilibrium with endogenous network formation where buyers initiate link formations is as follows:

Definition 2. *Given*

a collection of potential supplier sets for each firm, $\{\mathbf{Z}_j\}_{j\in\Omega}$*,*
common markups for sales to households, $\mu_{jH} = \mu_H = \frac{\sigma}{\sigma-1}\ \forall j \in \Omega$*,*
common markups for sales to firms, $\mu_{jk} = 1\ \forall j, k \in \Omega$*,*
an aggregate labor supply, L*, and*
a normalization of the wage level, w*,*

an ***endogenous network equilibrium with buyers initiating contacts*** *is characterized by*

firm-level costs, $\{c_j\}_{j\in\Omega}$*,*
firm-level labor demand, $\{\ell_j\}_{j\in\Omega}$*,*
quantities of goods purchased by households, $\{q_{jH}\}_{j\in\Omega}$*,*
quantities of goods purchased by firms, $\{q_{jk}\}_{j,k\in\Omega}$*,*
a price index for the consumer, P*,*
an aggregate expenditure, E*, and*
a set of sourcing strategies, $\{Z_j\}_{j\in\Omega}$*,*

such that firms solve the optimization problem characterized by (14) *and* (15)*, and equations* (2)*,* (3)*,* (4)*,* (5)*,* (7)*,* (8)*,* (9)*,* (10)*,* (11)*, and* (12) *hold.*

3.1.2 Equilibrium when suppliers initiate link formations

We now turn to an economy where only suppliers can initiate linkages with other firms. Suppliers choose the optimal set of buyers within their potential buyers and pay fixed costs in terms of labor that vary for each firm-pair. Upon establishing links, suppliers charge a monopolistic competitive markup on their sales to other firms; hence, all suppliers charge a markup of $\mu = \frac{\sigma}{\sigma-1}$ on all their sales. Therefore, variable profits defined in equation (10) are now $\pi_j^{var} = \frac{1}{\sigma}x_j$.

When suppliers initiate the network formation, they take as given their own set of suppliers and their own cost, the household demand shifter, $\frac{E}{P^{1-\sigma}}$, and also the demand shifter of their potential buyer firms, $\frac{x_k}{c_k^{1-\sigma}}$. These assumptions on firm behavior are the same as those made in other closely related supplier-initiated network formation models in the literature (e.g., Lim 2018, Huneeus 2018, and Bernard et al. 2022).

Given its own selling strategy, B_j, the profit of firm j is equal to its variable profit less the fixed costs of link formation, $\sum_{k\in B_j} f_{jk}w$:

$$\pi_j(B_j) = \frac{1}{\sigma}\left(\beta_{jH}^{\sigma-1}\mu^{1-\sigma}\phi_j^{\sigma-1}\Theta_j\frac{E}{P^{1-\sigma}} + \sum_{k\in B_j}\alpha_{jk}^{\sigma-1}\mu^{1-\sigma}\phi_j^{\sigma-1}\Theta_j\frac{x_k/\mu}{\Theta_k}\right) - \sum_{k\in B_j} f_{jk}w. \tag{16}$$

We assume that firm j exogenously meets a set of potential buyers, $\mathbf{B}_j$. Firm j then endogenously decides on the set of buyers:

$$\max_{B_j}\ \pi_j(B_j) \quad \text{s.t.} \quad B_j \subseteq \mathbf{B}_j. \tag{17}$$

As shown in equation (16), given the demand shifters of firm j's potential buyers in $\mathbf{B}_j$, the marginal benefit and marginal cost of adding a firm $k \in \mathbf{B}_j$ as a buyer are additively separable from those of adding other potential buyers. Therefore, the firm is able to choose whether to supply each potential buyer independent of other buyers.

We now define the equilibrium with endogenous network formation and suppliers initiating contacts.

Definition 3. *Given*

a collection of potential buyer sets for each firm, $\{\mathbf{B}_j\}_{j\in\Omega}$,
common markups for sales to households, $\mu_{jH} = \mu_H = \frac{\sigma}{\sigma-1}\ \forall j \in \Omega$,
common markups for sales to firms, $\mu_{jk} = \frac{\sigma}{\sigma-1}\ \forall j,k \in \Omega$,
an aggregate labor supply, L, *and*
a normalization of the wage level, w,

an ***endogenous network equilibrium with suppliers initiating contacts*** *is characterized by*

firm-level costs, $\{c_j\}_{j\in\Omega}$,
firm-level labor demand, $\{\ell_j\}_{j\in\Omega}$,
quantities of goods purchased by households, $\{q_{jH}\}_{j\in\Omega}$,
quantities of goods purchased by firms, $\{q_{jk}\}_{j,k\in\Omega}$,
a price index for the consumer, P,
an aggregate expenditure, E, *and*
a set of selling strategies, $\{B_j\}_{j\in\Omega}$,

such that firms solve the optimization problem characterized by (16) *and* (17), *and equations* (2), (3), (4), (5), (7), (8), (9), (10), (11), *and* (12) *hold.*

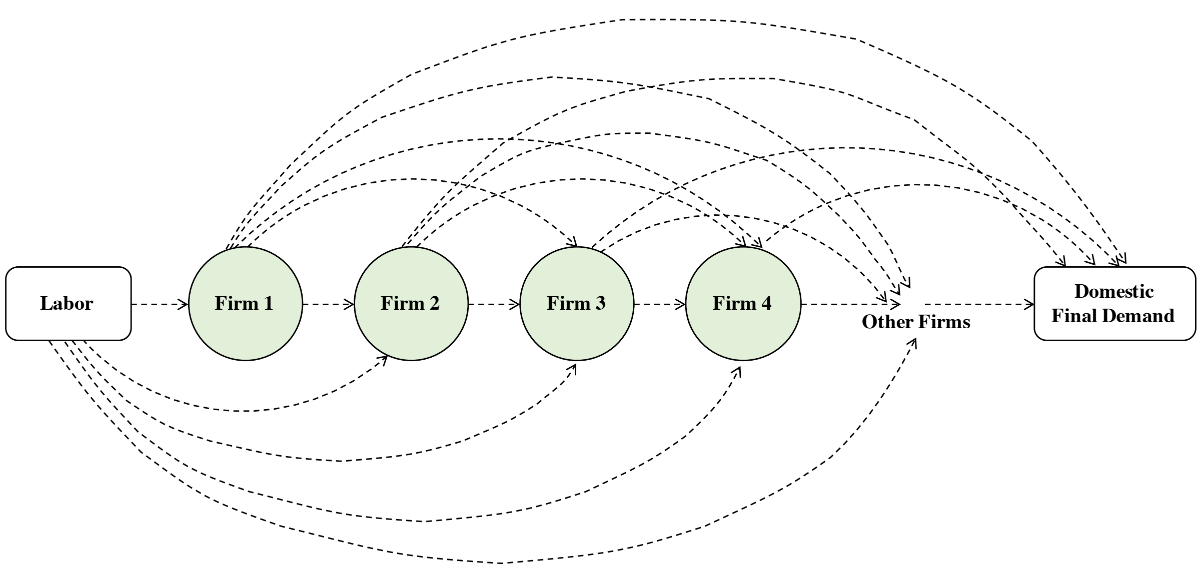
3.2 Uniqueness versus Multiplicity of Endogenous Network Formation Equilibria

We next discuss results for the uniqueness of the endogenous network formation equilibrium when the buyers initiate the link formation and all feasible production networks are acyclic. We then provide examples of multiple equilibria when these conditions are not met (e.g., when there are cycles in the feasible networks or when the supplier initiates the link formation).

3.2.1 Uniqueness and existence of equilibrium for buyer-initiated endogenous network formation under acyclic production networks

We first describe how the equilibrium can be computed before making a constructive argument that the equilibrium under endogenous network formation is unique when the buyer initiates the link formation and the set of feasible production networks is acyclic. We postulate an ordering of firms and restrict the potential set of suppliers to firms that appear prior to the buyer. Specifically, we order firms in a sequence $S = \{1, 2, 3, ..., N\}$ that restricts the set of potential suppliers, $\mathbf{Z}_j$, to be the firms prior in the sequence.[9] The structure of an acyclic network is depicted in Figure 1. Because we assume that buyers have full bargaining power over markup setting in any firm-to-firm transactions, a firm's profitability is affected only by the sourcing strategies, hence the marginal costs, of their potential suppliers as well as by the domestic household demand, $A \equiv \frac{E}{P^{1-\sigma}}$, which it takes as given. This allows us to solve the problem of firms sequentially given a guess for domestic household demand. In the network depicted in Figure 1, firm 1 is the first in the ordering, and it can only hire labor inputs. It makes its labor hiring decision based on the guess of domestic household demand, A, and the wage, w (which is normalized). Firm 2—the firm next in the ordering—can hire labor and also decide whether to purchase from firm 1. It makes these decisions based on the domestic household demand, the wage, and the cost of its potential supplier (firm 1). Firm 3 makes its decision based additionally on firm 2's cost, and so on. This solution algorithm, defined more formally in Appendix A.6.1, combined with a tie-breaking rule in case a firm is indifferent between several sourcing strategies, implies that for a given level of domestic household demand, A, the firms' sourcing strategies and costs are unique.

Figure 1: Endogenous acyclic network formation: potential connections



To show the uniqueness of the equilibrium, we first note that an equilibrium is a fixed point of domestic household demand satisfying all equilibrium conditions. In other words, an initial guess

[9] This is equivalent to restricting the technology parameters to satisfy $\alpha_{jk} = 0 \ \forall k < j$.

for domestic household demand, A_0, is an equilibrium if the implied domestic household demand, A_1, determined by steps 1-3 of the solution algorithm in Appendix A.6.1, satisfies $A_1 = A_0$. With a slight abuse of notation, we denote the implied domestic household demand $A_1(A_0)$ as a function of the initial guess A_0. Therefore, the uniqueness of the equilibrium can be established by examining the shape of $A_1(A_0) - A_0$ as a function of A_0. Specifically, we show that $A_1(A_0) - A_0$ is strictly decreasing in A_0, which implies that if an equilibrium exists, it is also unique.

We start by presenting two lemmas that characterize the sourcing strategies, sourcing capabilities, and marginal costs of firms in response to an increase in domestic household demand. Lemma 1 states that given the sourcing strategies of potential suppliers, expanding firm j's sourcing strategy improves its sourcing capability and lowers its marginal cost. Lemma 2 states that the sourcing strategies of all firms are weakly expanding in response to greater household demand.

Lemma 1. *For any firm $j \in \Omega$, given the sourcing strategies of firm j's potential suppliers, if $Z_j \subset Z'_j$, then $\Theta_j(Z'_j) \geq \Theta_j(Z_j)$ and $c_j(Z'_j) \leq c_j(Z_j)$.*

Proof. See Appendix A.2. □

Lemma 2. *When $A' > A$, $Z_j(A) \subset Z_j(A')$ for all $j \in \Omega$.*

Proof. See Appendix A.2. □

The intuition behind the two lemmas is that when domestic household demand increases, the benefit of including a potential supplier in a firm's sourcing strategy increases as well. First, greater domestic household demand raises the variable profits given firm's current sourcing strategy. Second, if an upstream firm j adds a new supplier to its sourcing strategy, its marginal cost decreases as a result of its CES production function, and this raises the marginal benefits of using firm j's output for downstream firms. Since the fixed costs of link formation do not change, firms keep all existing links and add a new supplier whenever the net benefit of doing so is non-negative. Therefore, the marginal costs of all firms weakly decrease (strictly decrease if a firm or its suppliers form a new link), and the sourcing capabilities of all firms weakly increase upon an increase in domestic household demand.

We now show in Proposition 1 that $A_1(A_0) - A_0$ is decreasing in A_0 when all firms charge the same markup when selling to final consumers.

Proposition 1. *Suppose buyers initiate link formations and have full bargaining power over markup setting. Assume all firms charge the same markup when selling to final consumers: $\mu_{jH} = \mu_H$ for all $j \in \Omega$. In addition, assume that all feasible networks are acyclic. Then,*

$$A_1(A_0) - A_0 = -\frac{1}{\mu_H} A_0 + \frac{wL - \sum_{j\in\Omega}\sum_{k\in Z_j(A_0)} f_{kj} w}{\mu_H^{1-\sigma}\sum_{j\in\Omega}\beta_{jH}^{\sigma-1}\phi_j^{\sigma-1}\Theta_j(Z_j(A_0))} \tag{18}$$

is decreasing in A_0.

Proof. See Appendix A.2. □

The basic intuition for Proposition 1 is that Lemmas 1 and 2 say that both sourcing capabilities and fixed costs increase as domestic household demand A_0 increases. This pushes down the second term on the right hand side of equation (18). Proposition 1 has two implications. First, if an equilibrium domestic household demand A exists, it is also unique. Second, when $\mu_H > 1$, the slope of $A_1(A_0) - A_0$ is between -1 and 0. This implies that the equilibrium A, if it exists, can be achieved starting from any initial guess A_0. Note that the uniqueness of A implies that the equilibrium in Definition 2 is unique. Without loss of generality, solving the sourcing problem of firms sequentially—following the ordering in S—yields a unique vector of firm-level costs and aggregate expenditure. Following the procedures described in Section 2.5, other equilibrium variables are uniquely determined.

We next turn to the existence of an equilibrium. Since link formation is essentially a discrete choice problem, the implied domestic household demand $A_1(A_0)$ may not continuously adjust when changes occur in the network. Therefore, it is possible that an equilibrium does not exist. To ensure both the existence and uniqueness of the equilibrium, we envision each type of firm as consisting of a continuum of firms, each solving the problem described above, but with the discrete choice of whether or not to include a supplier augmented by a draw from the logit distribution (see Train 2009). Integrating over the continuum of firms within each type then leads the firm type's sourcing strategy to adjust continuously in response to changes in domestic household demand, which guarantees the existence of an equilibrium. Note that if the dispersion parameter of the logit distribution is infinitesimal, then the implied domestic household demand $A_1(A_0)$ converges to the one with discrete adjustments where all firms in each firm type are identical. We discuss this logit smoothing methodology in detail in Appendix A.3.

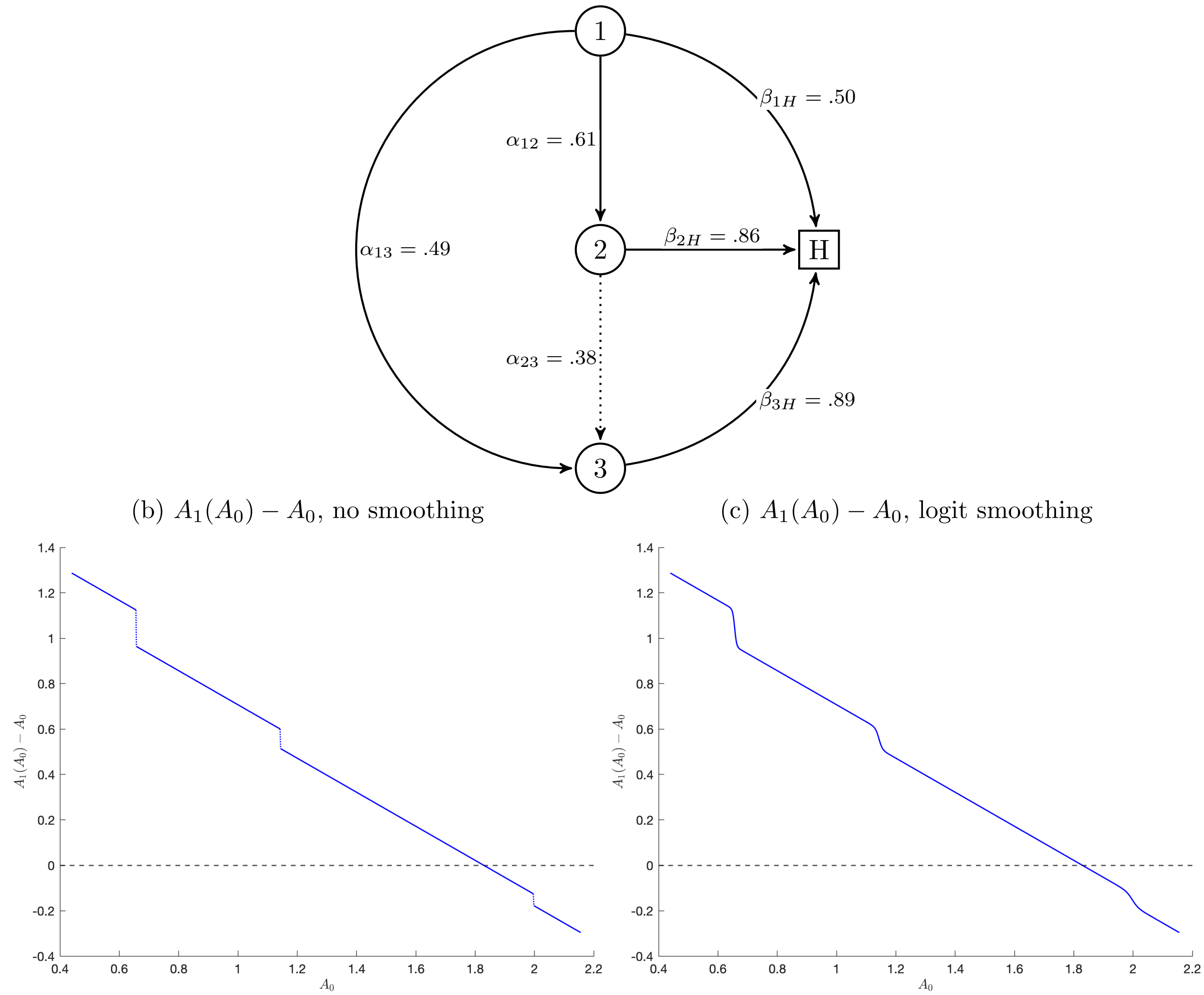
We illustrate the uniqueness and existence property of the equilibrium using an example of three firms. Specifically, we create a simple economy with three firms and plot $A_1(A_0) - A_0$ as a function of A_0 (equation (18)), which is informative about whether an equilibrium exists or is unique.[10] We plot the technology and preferences of the three-firm economy in Figure 2a and $A_1(A_0) - A_0$ in Figures 2b and 2c.[11] We observe a pattern as expected from the results in Proposition 1: $A_1(A_0) - A_0$ is decreasing in A_0. Each discontinuity represents a change in network structure after a small increase in A_0. The equilibrium domestic household demand is characterized by the A_0 that satisfy $A_1(A_0) - A_0 = 0$ and is unique. However, under different parameter values, it is possible that an equilibrium does not exist if $A_1(A_0) - A_0$ fails to cross the horizontal line of zero. When we apply the method of logit smoothing, as shown in Figure 2c, $A_1(A_0) - A_0$ is now continuous in

[10] Throughout the examples, we let $\sigma = 4$, $L = 1$, and normalize the domestic wage, $w = 1$. The relative importance of labor input in production is normalized to be one for all firms: $\alpha_{Lj} = 1$ for all j. We assume that the productivity levels for all firms are one (i.e., $\phi_j = 1$ for all j). For each of the four examples described later we choose α_{jk} and β_{jH} that satisfy the conditions discussed in Section 2.5, so that the equilibrium is unique once given a network structure.

[11] We plot the range of domestic household demand given by $[A_{min}, A_{max}]$. Specifically, the lower bound A_{min} is achieved when all firms are connected with each other, in which case the marginal costs are the lowest, and firms sell at marginal cost and make no profit ($E = wL$). On the other hand, the upper bound A_{max} is achieved when no link is formed, in which case the marginal costs are the highest, and firms charge a monopolistically competitive markup $\frac{\sigma}{\sigma-1}$ ($E = \frac{\sigma}{\sigma-1}wL$). Note that $[A_{min}, A_{max}]$ contains all feasible domestic household demands. Hence, any equilibrium domestic household demand lies within this range.

Figure 2: Three-firm example: endogenous acyclic network with buyers initiating contacts

(a) Technology and preferences



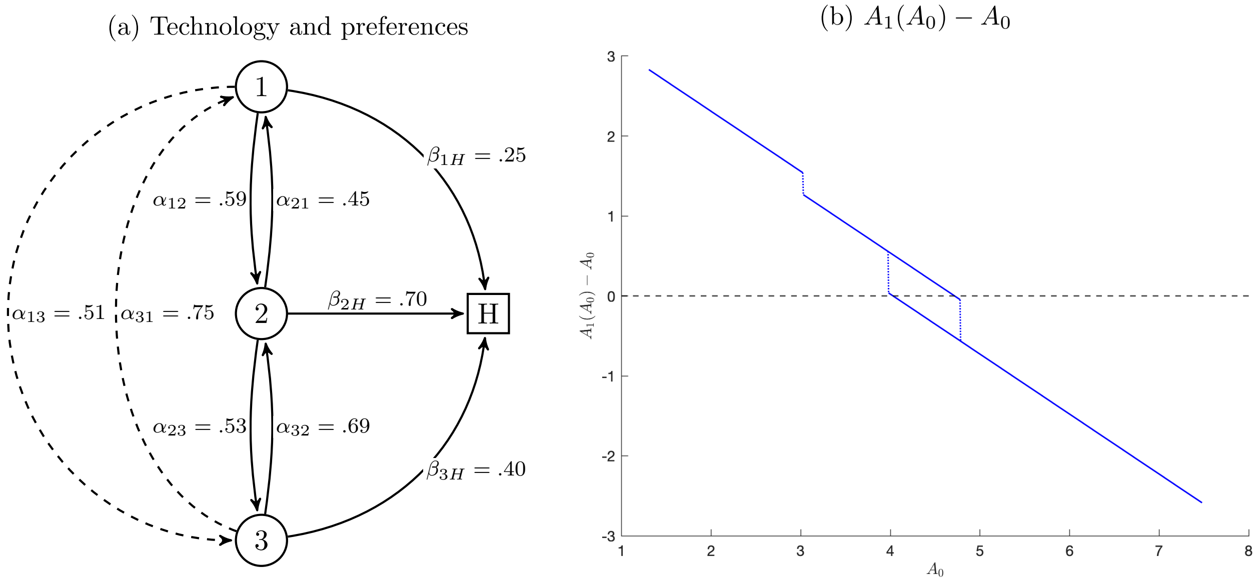
Notes: The figure illustrates an example of three firms in which a unique equilibrium exists under an acyclic network with buyers initiating contacts. Panel (a) displays the feasible network structure and the values for α_{jk} and β_{jH} for all j, k. Feasible links that are formed in equilibrium are indicated by the solid lines. Feasible links that are not formed in equilibrium are indicated by the dotted lines. Panel (b) plots $A_1(A_0) - A_0$ as a function of A_0 based on equation (18). The unique equilibrium domestic household demand is $A = 1.83$. Panel (c) is analogous to Panel (b) but applies the method of logit smoothing. The computation under logit smoothing for Panel (c) is described in Appendix A.3. The logit dispersion parameter is 0.00005. The values of parameters not shown in Panel (a) are as follows: $\sigma = 4$, $L = 1$, $\phi_j = 1$, $\alpha_{Lj} = 1$, $f_{jk} = 0.01$ for all j, k.

A_0, implying that an equilibrium exists and is unique.

3.2.2 Example of multiple equilibria when the network formation is buyer initiated and the production network is cyclic

We next turn to a more general network structure that allows for cycles in the production network. In cyclic production networks, the marginal cost of a firm's supplier is potentially dependent on the marginal cost of the firm itself. This setup may lead to multiple equilibria due to coordination failure. On the one hand, if a potential supplier of a firm has a high marginal cost the firm may not decide to source from that potential supplier, resulting in the firm having a high marginal cost. Because of this, the potential supplier—who is also downstream from the firm—may not decide to source from the firm, which would raise the marginal cost of the potential supplier. On the other hand, if the potential supplier has a low marginal cost the firm may decide to source from that potential supplier, resulting in the firm having a low marginal cost. Because of this, the potential supplier may source from the firm, which in turn would lower the marginal cost of the potential supplier.

Figure 3: Three-firm example: endogenous cyclic network with buyers initiating contacts



Notes: The figure illustrates an example of three firms in which there exist two equilibria under a cyclic network with buyers initiating contacts. Panel (a) displays the feasible network structure and the values for α_{jk} and β_{jH} for all j, k. Feasible links that are formed in both equilibria are indicated by the solid lines. Feasible links that are formed in one of the two equilibria are indicated by the dashed lines. Panel (b) plots $A_1(A_0) - A_0$ as a correspondence of A_0 based on equation (18). In the first equilibrium, equilibrium domestic household demand is $A = 4.04$, and the following links are formed: $1 \rightleftarrows 2$, $2 \rightleftarrows 3$, $1 \rightleftarrows 3$. In the second equilibrium, equilibrium domestic household demand is $A = 4.72$, and the following links are formed: $1 \rightleftarrows 2$, $2 \rightleftarrows 3$. The values of parameters not shown in Panel (a) are as follows: $\sigma = 4$, $L = 1$, $\phi_j = 1$, $\alpha_{Lj} = 1$ for all j, and $f_{21} = 0.0001$, $f_{31} = 0.0043$, $f_{12} = 0.0052$, $f_{32} = 0.0020$, $f_{13} = 0.0052$, $f_{23} = 0.0050$.

The key distinction from the case in which the network is acyclic is that here firms' decisions depend on what they take as given for both household demand and the whole network structure between firms. Note that not all feasible networks are consistent with a given guess for household demand. This occurs when at least one firm has an incentive to deviate from the set of suppliers implied by the network structure, given the level of household demand and strategies for the set of suppliers by all other firms. The function $A_1(A_0)$ is a single valued function when the network is acyclic, but here it is a correspondence that can take multiple values. Figure 3 shows an example of multiple equilibria with buyers initiating contacts and a cyclic network. Figure 3a shows all possible links and assigned values for α_{jk} and β_{jH} in a simple example with three firm types. Note that in this cyclic network, downstream firms are allowed to sell to upstream suppliers. For each A_0, we compute the implied A_1 for all possible networks that are consistent with A_0. As shown in Figure 3b, the correspondence $A_1(A_0) - A_0$ crosses zero at two different points, which implies that multiple equilibria exist.

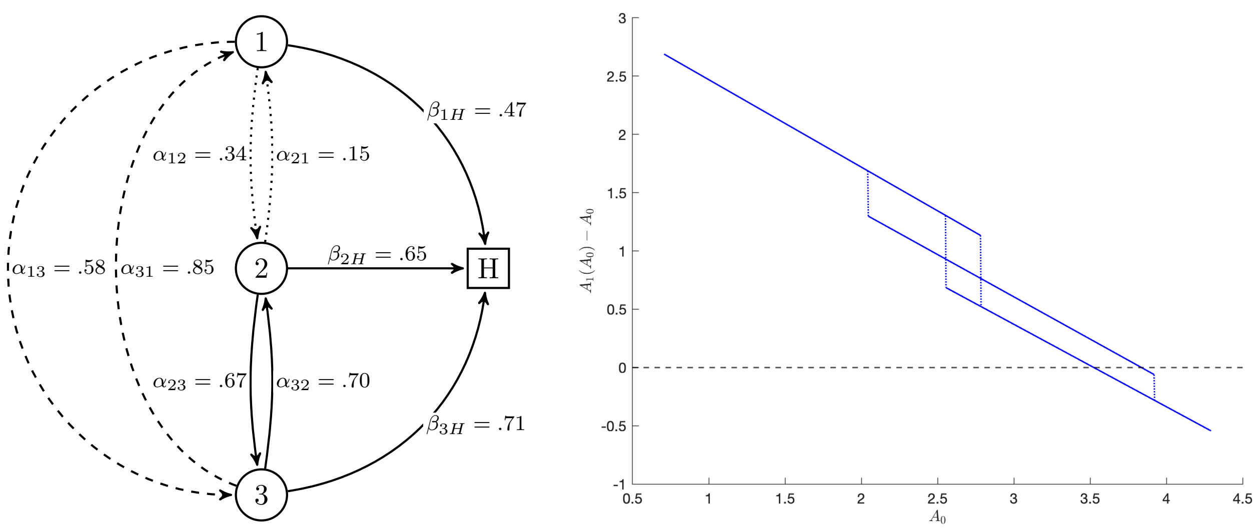
We note that in the three-firm example, parameter values are chosen such that the coefficient matrices in equations (4) and (8) are invertible for any feasible network. This implies that, given a feasible network, both the cost vector and the sales vector are uniquely determined. As discussed in Section 2.5, the fixed network equilibrium is unique. Therefore, the presence of multiple equilibria under endogenous network formation is solely attributed to multiplicity in network, rather than multiple equilibria within a fixed network. Moreover, in the following sections where suppliers initiate contacts, we also ensure that the equilibrium within a fixed network is unique for the chosen parameterization.

3.2.3 Example of multiple equilibria when the network formation is supplier initiated and the production network is cyclic

Figure 4: Three-firm example: endogenous cyclic network with suppliers initiating contacts

(a) Technology and preferences

(b) $A_1(A_0) - A_0$



Notes: The figure illustrates an example of three firms in which there exist two equilibria under a cyclic network with suppliers initiating contacts. Panel (a) displays the feasible network structure and the values for α_{jk} and β_{jH} for all j, k. Feasible links that are formed in both equilibria are indicated by the solid lines. Feasible links that are formed in one of the two equilibria are indicated by the dashed lines. Feasible links that are not formed in either equilibrium are indicated by the dotted lines. Panel (b) plots $A_1(A_0) - A_0$ as a correspondence of A_0 based on equation (18). In the first equilibrium, equilibrium domestic household demand is $A = 3.52$, and the following links are formed: $2 \rightleftarrows 3$, $1 \rightleftarrows 3$. In the second equilibrium, equilibrium domestic household demand is $A = 3.84$, and the following links are formed: $2 \rightleftarrows 3$. The values of parameters not shown in Panel (a) are as follows: $\sigma = 4$, $L = 1$, $\phi_j = 1$, $\alpha_{Lj} = 1$, $f_{jk} = 0.01$ for all j, k.

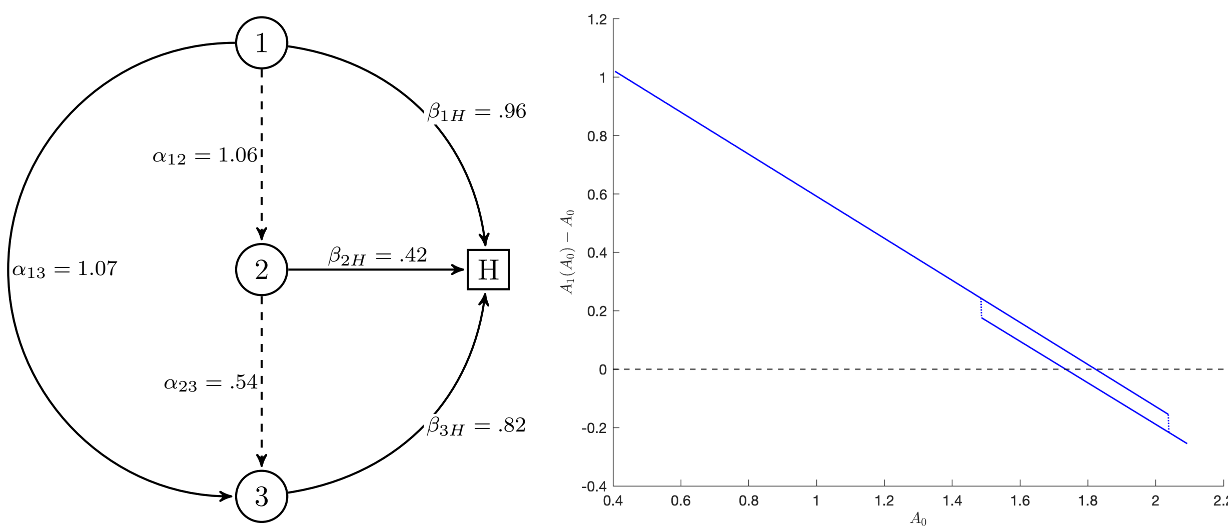
We now turn to endogenous network formation when suppliers initiate link formations. Here, a source of multiplicity is that a firm's choice of its set of buyers depends not only on household demand but also on the demand it faces from other firms. On the one hand, if a firm faces a potential buyer with a high demand level, it may decide to sell to the potential buyer. This lowers the cost of the potential buyer, which in turn increases its demand for the firm's product. On the other hand, if a firm faces a potential buyer with a low demand level, it may not decide to sell to the potential buyer. This increases the cost of the potential buyer, resulting in a low demand for the firm's product.

As in the previous section, a firm's decision depends on the level of domestic household demand, A, and the entire network structure, which affects the scale of its buyers. A network may not be consistent with a given level of domestic household demand when at least one firm has an incentive to deviate from the set of buyers implied by the network structure, given the strategies for the set of buyers by all other firms. Figure 4 depicts an example of suppliers initiating contacts in a cyclic network. The figure shows two possible levels of domestic household demand, each constituting an equilibrium.

3.2.4 Example of multiple equilibria when the network formation is supplier initiated and the production network is acyclic

Figure 5: Three-firm example: endogenous acyclic network with suppliers initiating contacts

(a) Technology and preferences | (b) $A_1(A_0) - A_0$



Notes: The figure illustrates an example of three firms in which there exist two equilibria under an acyclic network with suppliers initiating contacts. Panel (a) displays the feasible network structure and the values for α_{jk} and β_{jH} for all j, k. Feasible links that are formed in both equilibria are indicated by the solid lines. Feasible links that are formed in one of the two equilibria are indicated by the dashed lines. Panel (b) plots $A_1(A_0) - A_0$ as a correspondence of A_0 based on equation (18). In the first equilibrium, equilibrium domestic household demand is $A = 1.73$, and the following links are formed: $1 \to 2$, $2 \to 3$, $1 \to 3$. In the second equilibrium, equilibrium domestic household demand is $A = 1.82$, and only link $1 \to 3$ is formed. The values of parameters not shown in Panel (a) are as follows: $\sigma = 4$, $L = 1$, $\phi_j = 1$, $\alpha_{Lj} = 1$, and $f_{jk} = 0.006$ for all j, k.

Note that imposing network acyclicity does not guarantee equilibrium uniqueness when suppliers initiate link formations. Since firms still generate profits by selling to other firms, their choices of their set of buyers depend on the demand from other firms in addition to household demand. The source of multiplicity in the equilibrium remains as the demand from potential buyers is a function of the firm's selling strategy. Figure 5 shows an example of multiple equilibria with suppliers initiating contacts in an acyclic network.

3.2.5 Summary

As summarized in Table 2, we find multiple equilibria when the buyers initiate the network linkages and production networks are cyclic or when the suppliers initiate the network linkages (regardless of whether the network is acyclic or cyclic). We note that these multiple equilibria are different from each other not only in terms of the identity of the connections, but also in terms of their economic outcomes, such as differences in the number of connections and in aggregate real income. In Appendix A.4, we illustrate that in response to the same change in fixed costs, the number of connections and real income may respond in different magnitudes depending on which equilibrium is realized.

Table 2: Guaranteed uniqueness of equilibrium

	Contact initiator	
Network structure	Buyer	Supplier
Acyclic	✓ (Proposition 1)	✗ (Figure 5)
Cyclic	✗ (Figure 3)	✗ (Figure 4)

For the remainder of the paper, because of the greater tractability and the desirable uniqueness property, we will proceed with the model in which all assumptions specified in Proposition 1 hold: buyers initiate the network linkages, production networks are acyclic, and all firms charge the same markup when selling to final consumers.

3.3 Extension to a Small Open Economy

Our model of endogenous network formation has so far restricted attention to closed economies. The model can, however, be adapted to describe a small open economy by adding a foreign supplier and a foreign buyer. We add the foreign suppliers and foreign buyers in the same way as what was discussed in Section 2.6. Specifically, we add the foreign supplier in each domestic firm's set of potential suppliers. We assume that all domestic firms take the import prices, $\{p_{Fj}\}_{j\in\Omega}$, as given. For exports, firms take as given the export costs, $\{\tau_{jF}\}_{j\in\Omega}$, and the foreign demand shifter, D_F. We refer to the detailed description of the small open economy model in Appendix A.5.

With buyers initiating contacts and an acyclic domestic production network, the firms' problem can be solved sequentially given import prices, foreign demand, and guesses for domestic household demand and the wage level.[12] Firms pay fixed costs of f_{Fj} when importing and f_{jF} when exporting.

[12] Also possible is that cycles involving a foreign trading partner appear. For example, consider a network in which

Conditional on a binary choice of whether or not to export, a firm's decision to form a link with a supplier remains independent across all of its feasible suppliers. It is therefore natural to solve the firm-level problem of choosing domestic suppliers, import participation, and export participation in two steps: in an outer problem, the firm decides on the export participation and in an inner problem, the firm selects the domestic suppliers and import participation (taking as given the export participation in the inner problem).

Note that once the model is extended to a small open economy, in addition to the domestic household demand, the domestic wage enters as an additional equilibrium variable that can no longer be normalized. Therefore, our uniqueness result for the closed economy does not directly apply to the small open economy. However, one can check numerically whether the equilibrium is likely to be unique. Note that the sourcing strategies are unique given the domestic household demand, A, and the wage level, w. So the question is whether there are multiple combinations of domestic household demand and wage that solve all equilibrium conditions. To check this, one can numerically plot the norm of the difference between the initial guesses of the two variables (w_0, A_0) and their implied values (w_1, A_1). The equilibrium is unique when there is a global minimum that attains the smallest norm of zero. We visualize that for our estimated parameters in Section 4.3, there likely exists a unique equilibrium to the small open economy. We describe the methodology and results in detail in Appendix A.5.

4 Data and Quantification

4.1 Data Sources and Sample Selection

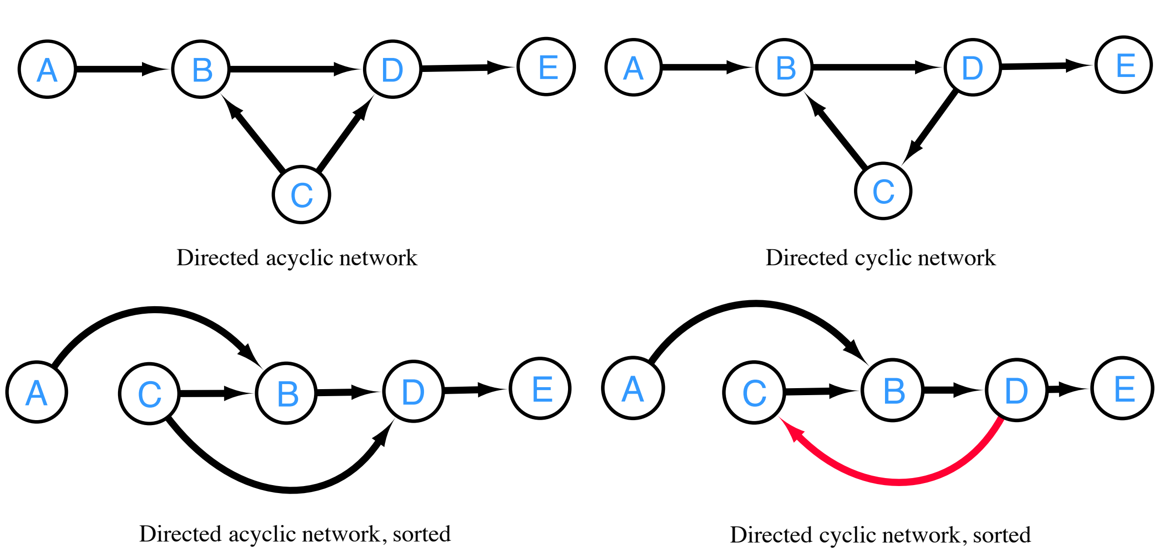
Our analyses draw on three administrative data sources from Belgium for the period 2002-2014: the Business-to-Business (B2B) transactions database, the annual account data, as well as the Belgian customs records and the intra-EU trade declarations. These data sources can be linked through unique identifiers, assigned and recorded by the government for the purpose of collecting value-added taxes (VAT). As a firm may have multiple VAT identifiers, we aggregate the data up to the firm level using information from the balance sheets about ownership structure. Then, we restrict our analysis to firms in the private and non-financial sectors with positive labor costs, at least one full-time-equivalent employee, and positive output. Following de Loecker, Fuss, and Van Biesebroeck (2014), we also restrict our analysis to firms with tangible assets of more than 100 euro and positive total assets in at least one year during our sample period. After applying these restrictions, our sample consists of 98,745 firms in 2012, which is the main year of our analysis for this paper. Section 2 of Dhyne, Kikkawa, Mogstad, and Tintelnot (2021) describes the data sources and sample selection in detail.

firm A exports, firm B imports, and firm B supplies to firm A. However, since we assume a small open economy in which both import prices and foreign demand shifters are fixed, this cycle involving a foreign trading partner does not materially affect the solution to firm A's and B's problems.

4.2 Assessing Network Acyclicity

As discussed in Section 3, we focus on the economy with endogenous network formation where the assumptions specified in Proposition 1 hold. In particular, we consider a production network that is acyclic. As shown in Figure 6, in an acyclic firm network, there is at least one way to sort firms so that all directed edges face one direction. In contrast, in a cyclic network, at least one edge will face the opposite direction. This feature of our network formation mechanism is admittedly restrictive. We now perform two checks to assess how well the Belgian data can be approximated by an acyclic network.

Figure 6: Examples of acyclic and cyclic networks



4.2.1 How cyclic is the production network?

Let $\nu(\cdot)$ be an ordering of firms that maps firm $i \in \Omega$ into numbers from $\{1, \cdots, N\}$. To describe how cyclic the Belgian production network is, we want to find the optimal $\nu(\cdot)$ that minimizes the following objective function:

$$\min_{\{\nu(\cdot)\}} \sum_{i,j \in \Omega} \mathbb{I}\{i \in Z_j\} \mathbb{I}\{\nu(i) > \nu(j)\},$$

where Z_j is the supplier set of firm j. Solving this problem corresponds to minimizing the number of links that violate network acyclicity.

To solve this problem, which is also known as the feedback arc set problem, we adopt an algorithm proposed by Eades, Lin, and Smyth (1993). The details of the computational algorithm and implementation are presented in Appendix B. Intuitively, the algorithm places firms with a high net outdegree (number of outgoing links minus number of incoming links) at the beginning of the ordering and those firms with a low net outdegree toward the end of the ordering. The algorithm offers a local minimum, showing that at most, 18 percent of edges in the whole firm-to-firm network in 2012 violate acyclicity.[13] We also search for an ordering that minimizes the value of firm-to-firm sales in violation of acyclicity.[14] We find that no more than 23 percent of firm-to-firm sales are in violation of acyclicity. We will refer to the former as the unweighted ordering algorithm and the latter as the weighted ordering algorithm. Note that under both approaches, this heuristic algorithm does not guarantee a global minimum solution. It is likely that there is another ordering where the approximation of an acyclic network is even better than the one we find.[15]

A natural question that arises is how different the structure of an economy with an acyclic network is in comparison to the economy observed in the data. One way to make this comparison is to calculate input-output tables with and without the firms in buyer-supplier relationships that violate acyclicity. We find that when calculating input-output tables with 72 sectors, the correlations between the input-output table coefficients from the full data and the data without links in violation of the ordering are high.[16] When we use the unweighted ordering algorithm output, the correlation is 0.90, and it is even higher, 0.97, when using the weighted ordering algorithm output.[17]

4.2.2 Real income responses to export cost changes under fixed networks: cyclic versus acyclic production networks

Another way to assess the assumption of an acyclic network is to examine how the results based on the exogenous network model change if we exclude transactions that violate acyclicity. It is reassuring to find that the estimated effect of international trade on consumer prices and firms' costs of production is very similar if we only use firm-to-firm sales that are consistent with the

[13]Liu (2019) finds that the industry input-output matrices of South Korea and China are also approximately acyclic once small entries are removed. Similar to Liu (2019), we also find that much of the trade flows that violate acyclicity consist of small transactions from the perspective of the supplier. Once we drop trade flows that account for less than 5 percent of the supplier's total sales to other Belgian firms and run the unweighted ordering algorithm, we find an ordering in which only 6 percent of edges violate acyclicity. The weighted ordering algorithm offers an ordering in which only 11 percent of firm-to-firm sales are in violation of acyclicity. In a different context, Simpson, Srinivasan, and Thomo (2016) calculate that 23 percent of edges are in violation of acyclicity in the Twitter network in the year 2010.

[14]Specifically, we solve the following problem: $\min_{\{\nu(\cdot)\}} \sum_{i,j} x_{ij} \mathbb{I} \{\nu(i) > \nu(j)\}$, where x_{ij} is the value of the sales from firm i to firm j.

[15]Coleman and Wirth (2010) and Simpson et al. (2016) find that compared to other algorithms proposed in the computer science literature, the algorithm of Eades et al. (1993) does well in terms of speed and finding a low objective.

[16]To construct the input-output table coefficients, we aggregate firm-to-firm transactions within the supplying and buying sector. We note that this procedure differs from the national account definition of an input-output table. First, the rows and columns of our aggregated tables are referring to the main sectors of the buyers and suppliers, but these firms can also have a significant share of their production in other sectors. Second, in national account tables, the contribution of the wholesale and retail sectors to the production of the other goods only refer to the trade margin of retailers and wholesalers. In our data, the wholesale and retail sectors are accounted based on their total sales and total input consumption and not on their trade margin (Dhyne, Kikkawa, Mogstad, and Tintelnot, 2023).

[17]These correlations remain high when we drop the diagonal coefficients. With the unweighted ordering algorithm output, the correlation is 0.89, and with the weighted ordering algorithm output, the correlation stays at 0.97. We also find that these high correlations are robust when doing the same exercise at the four-digit level with 524 industries. The correlation is 0.86 when using the unweighted ordering algorithm output and 0.89 when using the weighted ordering algorithm output.

acyclic network obtained by the ordering algorithm described in Section 4.2.1. Specifically, we keep the direct import share of each firm the same as in the data, set all transactions in violation of the ordering to zero, and adjust all other domestic firm-to-firm input shares such that the share of each firm j's input purchases, $\sum_{i \in Z_j} s_{ij}$, is unchanged.[18] The results presented in Table C1 in Appendix C.1 show that the real income responses to export cost changes under an exogenous network are virtually identical if we only use the subset of transactions for which the domestic production network is acyclic.

Given these results, we will make the convenient assumption in the quantitative analysis below that the feasible production network is acyclic. We note that the analysis in Section 4.2.1 assesses how cyclic the *realized* Belgian network is, whereas the restriction in the theoretical model is about the *potential* network, which itself is unobserved.

4.3 Determining Model Parameters

When allowing for endogenous network formation, we are not able to analytically solve the model. Instead, we use the data to determine the model parameters and then provide numerical results for several counterfactual analyses. The goal of these numerical analyses is to draw inferences about how endogenous network formation may affect the gains-from-trade calculation in an economy that matches our data in important ways.

In the estimation of our model, we simulate 100,000 Belgian firms, approximately the same size as our sample in 2012. A firm is characterized by a set of potential suppliers that satisfies the ordering, a core productivity level, a vector of firm-pair-specific cost shifters, a foreign input cost shifter, a foreign demand shifter, a vector of fixed cost draws for all potential suppliers, and fixed costs of importing and exporting. In our baseline analysis, we set the value of σ to 4, a common choice in the literature (see for example, Antras, Fort, and Tintelnot, 2017 and Oberfield and Raval, 2021). We discuss the sensitivity of our results to the value for this parameter below. We normalize firms' labor productivity shifters, $\alpha_{Lj} = 1$, and firms' domestic final demand shifters, $\beta_{jH} = 1$.[19] Firms are positioned in an order from 1 to 100,000. The sets of up to 1,000 potential suppliers are drawn randomly such that the potential suppliers are positioned earlier than the buyer in the order of firms.

As a first step in the estimation, we recover the productivity distribution of firms (scaled by some general equilibrium objects) from the identity

$$x_{jH}^{1/(\sigma-1)} s_{Lj}^{1/(\sigma-1)} = \phi_j \frac{PE^{1/(\sigma-1)}}{\mu w}. \tag{19}$$

Observing all the terms on the left hand side enables us to estimate the distribution $\phi_j \frac{PE^{1/(\sigma-1)}}{\mu w}$. Af-

[18] The only exception is when there is no other domestic supplier of a firm, in which case in the data with only acyclic transactions the domestic firm-to-firm input share is set to zero.

[19] Note that α_{Lj} as well as α_{kj} ($\forall k \in Z_j$) enter multiplicatively with the inverse of firm productivity, $1/\phi_j$ in the cost function in equation (4). We therefore normalize $\alpha_{Lj} = 1$. As we have only data on revenues and costs (not separated by price and quantity), we normalize the domestic final demand shifter, $\beta_{jH} = 1$.

ter visually inspecting the distribution, we assume it is log-normal and estimate the scale parameter to be -0.68 and the dispersion parameter to be 0.56.

We next turn to the parameterization for the distributions of cost and demand shifters. We assume that firm-pair-specific random variables—α_{kj}, α_{Fj}, and β_{jF}—are independent draws from three log-normal distributions that share a common dispersion parameter, $\Phi^{\alpha,\beta}_{disp}$, and have different scale parameters, $\Phi^{\alpha_{\mathrm{dom}}}_{scale}$, $\Phi^{\alpha_F}_{scale}$, $\Phi^{\beta_F}_{scale}$, respectively.[20]

Similarly, the fixed cost draws for domestic purchases from other firms, imports, and exports are drawn independently from three log-normal distributions with scale parameters $\Phi^{f_{\mathrm{dom}}}_{scale}$, $\Phi^{f_{\mathrm{imp}}}_{scale}$, and $\Phi^{f_{\mathrm{exp}}}_{scale}$ and a common dispersion parameter, Φ^{f}_{disp}. Overall, there are eight parameters to be estimated.

We use a method of simulated moments to estimate our parameters. We target three sets of moments to match. The first set of moments describes the domestic sourcing patterns of firms. For example, we target the model to match the 50th, 75th, and 95th percentiles from the distribution of number of suppliers. Following the procedure used by Eaton, Kortum, and Kramarz (2011), we include in the first vector of moments generated by the model, $\hat{m}_1(\Phi)$, the fraction of firms falling into the the four bins cut by the three percentiles of the number of suppliers. We also target the distribution of buyers, share of labor costs in firms' costs, and the firm-to-firm input shares (conditional on observing trade between firms). Using the same procedure as above, we include the fraction of firms in the four bins cut by the three percentiles of the distribution (using as thresholds the percentiles observed in the data). This generates 16 elements in the vector $\hat{m}_1(\Phi)$.

The second set of moments characterizes firm-level patterns of imports and exports. We include in $\hat{m}_2(\Phi)$ the share of firms that import and export, respectively. We also include the fraction of firms falling into the bins cut by the 50th, 75th, and 95th percentiles of firms' direct import share (conditional on importing). The firm-level import share is defined in equation (A3) as s_{Fj} in Appendix A.5. Furthermore, we target the three percentiles of the total import share defined in Dhyne et al. (2021). A firm's total import share captures the share of firm's total costs that originates from abroad, either directly as imported inputs or indirectly through the firm's domestic suppliers. Firm j's total import share, s^{Total}_{Fj}, is defined recursively as $s^{Total}_{Fj} = s_{Fj} + \sum_{k\in \mathbf{Z}_j \setminus \{F\}} s_{kj} s^{Total}_{Fk}$. Similarly, on the export side, we target the three percentiles of the direct export shares (conditional on exporting) as well as the total export shares defined in Dhyne et al. (2021). A firm's direct export share, r_{jF}, is the share of direct exports in the firm's total sales. A firm's total export share, r^{Total}_{jF}, is the share of the firm's sales that are exported abroad either directly or indirectly through the firm's domestic buyers. Firm j's total export share, r^{Total}_{jF}, is defined recursively as $r^{Total}_{jF} = r_{jF} + \sum_{k\in \mathbf{B}_j\setminus\{F\}} r_{jk} r^{Total}_{kF}$, where r_{jk} is the share of firm j's revenue that is generated from its sales to firm k. There are 18 elements in the vector $\hat{m}_2(\Phi)$.

As a third set of moments, we include important aggregate targets such as the ratio of aggregate

[20] We simulate firm productivity by drawing $(\ln \phi_j + 0.68)/0.56$ from an i.i.d. standard normal distribution. Similarly, we randomly draw the standardized log of the firm-pair-specific random variable from an i.i.d. standard normal distribution. To ensure computational stability, we winsorize the standard normal draws at the 0.5th and 99.5th percentiles.

exports to aggregate final demand and the import content of domestic final demand. The import content of domestic final demand is defined as $\sum_{j\in\Omega} s_{jH} s_{Fj}^{Total}$, where s_{jH} is the share of aggregate expenditure E that is spent on firm j's good. There are 2 elements in the vector $\hat{m}_3(\Phi)$.

We describe the difference between the moments in the data and in the simulated model by $\hat{y}(\Phi)$:

$$\hat{y}(\Phi) = m - \hat{m}(\Phi) = \begin{bmatrix} m_1 - \hat{m}_1(\Phi) \\ m_2 - \hat{m}_2(\Phi) \\ m_3 - \hat{m}_3(\Phi) \end{bmatrix},$$

and the following moment condition is assumed to hold at the true parameter value Φ_0:

$$E\left[\hat{y}(\Phi_0)\right] = 0. \tag{20}$$

The method of simulated moments selects the model parameters that minimize the following objective function:

$$\hat{\Phi} = \arg\min_{\Phi} \left[\hat{y}(\Phi)\right]^{\top} \mathbf{W} \left[\hat{y}(\Phi)\right], \tag{21}$$

where $\mathbf{W}$ is a weighting matrix.[21]

4.3.1 Estimation Results

Table 3 shows the values of the estimated parameters. We note that the scale of the estimated parameters is affected by the choice of normalizations for the foreign market size and the price of the foreign input. We therefore focus on the model fit given these parameter estimates. Table 4 shows the targeted moments generated by the estimated parameters, compared with the moments from the data. Note that instead of showing the moments directly (i.e., the fraction of firms falling into each bin), we show the values of the 50th, 75th, and 95th percentiles in both the data and model. The estimated model fits well the targeted statistics of firm-to-firm transactions. In particular, the model succeeds in generating a more skewed distribution of the number of domestic buyers relative to the domestic suppliers. This enables the model to also match the differences in the distribution of total import shares and total export shares.

Table 3: Estimated parameters

Preference and production				Fixed costs			
$\hat{\Phi}^{\alpha_{\text{dom}}}_{scale}$	$\hat{\Phi}^{\alpha_F}_{scale}$	$\hat{\Phi}^{\beta_F}_{scale}$	$\hat{\Phi}^{\alpha,\beta}_{disp}$	$\hat{\Phi}^{f_{\text{dom}}}_{scale}$	$\hat{\Phi}^{f_{\text{imp}}}_{scale}$	$\hat{\Phi}^{f_{\text{exp}}}_{scale}$	$\hat{\Phi}^{f}_{disp}$
-3.20	-0.97	-1.15	0.93	-5.71	-2.94	-1.74	3.05

While we succeed in generating different distributions of firms' exposure to imports and exports, both directly and indirectly, the model also matches well the magnitude of aggregate imports relative to domestic final demand. Finally, we also match quite well the import content in domestic

[21] We weight the moments equally; hence, the weighting matrix is the identity matrix.

final demand, $\sum_{j\in\Omega} s_{jH} s_{Fj}^{Total}$, which is a key statistic that determines the magnitude of the change in the real wage in response to a change in the foreign price.

Table 4: Model fit: targeted moments

Moments	Data	Model
Number of dom. suppliers 50th percentile	33	25
Number of dom. suppliers 75th percentile	56	44
Number of dom. suppliers 95th percentile	141	95
Number of dom. buyers 50th percentile	9	15
Number of dom. buyers 75th percentile	35	40
Number of dom. buyers 95th percentile	184	131
Share of labor costs 50th percentile	0.34	0.35
Share of labor costs 75th percentile	0.54	0.52
Share of labor costs 95th percentile	0.84	0.85
Firm-to-firm input share 50th percentile	0.0012	0.0011
Firm-to-firm input share 75th percentile	0.0053	0.0068
Firm-to-firm input share 95th percentile	0.0443	0.0780
Share of firms that import	0.19	0.18
Direct import share (among importers) 50th percentile	0.28	0.16
Direct import share (among importers) 75th percentile	0.67	0.50
Direct import share (among importers) 95th percentile	0.89	0.92
Total import share 50th percentile	0.39	0.37
Total import share 75th percentile	0.55	0.50
Total import share 95th percentile	0.83	0.78
Share of firms that export	0.12	0.14
Direct export share (among exporters) 50th percentile	0.12	0.20
Direct export share (among exporters) 75th percentile	0.66	0.57
Direct export share (among exporters) 95th percentile	0.98	0.93
Total export share 50th percentile	0.03	0.09
Total export share 75th percentile	0.17	0.27
Total export share 95th percentile	0.72	0.63
Ratio of aggregate exports to aggregate sales to domestic final demand	0.82	0.84
Import content of domestic final demand	0.58	0.60

Notes: The percentiles for number of domestic suppliers and number of domestic buyers are calculated based on all firms in the sample. Share of labor costs refers to the fraction of labor costs in costs (labor costs+domestic purchases+imports), and the percentiles are calculated based on all firms in the sample. Firm-to-firm share refers to the fraction of costs a firm spends on one particular supplier, and the percentiles are calculated for all firm-to-firm transactions. The percentiles for direct import share are calculated for all firms with positive import purchases. Total import share is defined as $s_{Fj}^{Total} = s_{Fj} + \sum_{k\in\mathbf{Z}_j\setminus\{F\}} s_{kj} s_{Fk}^{Total}$, and the percentiles are calculated based on all firms in the sample. The percentiles for direct export share are calculated for all firms with positive export sales. Total export share is defined as $r_{jF}^{Total} = r_{jF} + \sum_{k\in\mathbf{B}_j\setminus\{F\}} r_{jk} r_{kF}^{Total}$, and the percentiles are calculated based on all firms in the sample. The import content of domestic final demand is defined as $\sum_{j\in\Omega} s_{jH} s_{Fj}^{Total}$.

We also examine how well the model fits moments that were not directly targeted in the estimation. The first set of rows in Table 5 report the size correlations between buyers and suppliers that trade with each other. Consistent with the data, the model predicts a weak negative correlation between the number of suppliers of the buying firms (indegree buyer) and the number of buyer firms of suppliers (outdegree supplier). Similarly, the correlation between sales of the buying and selling firms is close to zero in both the data and the model. From the second set of rows, we report various concentration measures captured by the shares of activities that the top 1 percent, 5 percent, or 10 percent firms account for. They show that our model is able to capture the high concentration in various sales and input measures. Finally, the last set of rows show the sales concentration at the level of firm-to-firm relationships. Both the data and the model predict that the top buyer-supplier pairs account for a large share of total firm-to-firm sales in the economy.

Table 5: Model fit: non-targeted moments

Moments	Data	Model
Corr (Indegree buyer, Outdegree supplier)	-0.05	-0.11
Corr (Sales buyer, Sales supplier)	-0.02	-0.00
Share of imports, top 1 percent of importers	0.66	0.59
Share of imports, top 5 percent of importers	0.83	0.83
Share of imports, top 10 percent of importers	0.90	0.91
Share of exports, top 1 percent of exporters	0.66	0.48
Share of exports, top 5 percent of exporters	0.84	0.77
Share of exports, top 10 percent of exporters	0.91	0.87
Share of labor costs, top 1 percent of firms	0.59	0.36
Share of labor costs, top 5 percent of firms	0.76	0.64
Share of labor costs, top 10 percent of firms	0.83	0.77
Share of network purchases, top 1 percent of firms	0.58	0.41
Share of network purchases, top 5 percent of firms	0.77	0.71
Share of network purchases, top 10 percent of firms	0.85	0.83
Share of network sales, top 1 percent of firms	0.62	0.54
Share of network sales, top 5 percent of firms	0.80	0.81
Share of network sales, top 10 percent of firms	0.88	0.90
Share of sales to dom. final demand, top 1 percent of firms	0.67	0.28
Share of sales to dom. final demand, top 5 percent of firms	0.80	0.58
Share of sales to dom. final demand, top 10 percent of firms	0.85	0.73
Share of total sales, top 1 percent of firms	0.68	0.45
Share of total sales, top 5 percent of firms	0.83	0.72
Share of total sales, top 10 percent of firms	0.88	0.83
Share of firm-to-firm sales, top 1 percent of links	0.69	0.64
Share of firm-to-firm sales, top 5 percent of links	0.86	0.85
Share of firm-to-firm sales, top 10 percent of links	0.91	0.92

Notes: The first row shows the correlation between the number of suppliers of the buying firms (indegree buyer) and the number of buyer firms of suppliers (outdegree supplier). The second row shows the correlation between sales of the buying and selling firms. For the rest, "Share of imports, top 1 percent of importers," for example, implies the share of imports by top 1 percent of importers among all importers.

We show that the equilibrium is unique given the parameters listed in Table 3. As described in

Section 3.3, we plot the norm of the difference between the initial guess (w_0, A_0) and the implied value (w_1, A_1). Figure A7 in Appendix A.5 suggests that there exists a global minimum that attains the smallest norm, implying that the equilibrium is unique.

4.4 Counterfactual Analyses

Equipped with the parameter estimates of our model, we explore the implications of allowing for endogenous networks in how the economy responds to foreign trade shocks. With the model where buyers initiate contacts in acyclic networks, we first consider a change in the costs of exporting, τ_{jF}, that only a subset of firms face when exporting abroad or a change in the import price, p_{Fj}, that only a subset of firms face when importing from abroad. We compute how equilibrium variables such as firm-level costs, sales, and the number of suppliers and buyers change in response to these foreign shocks, and illustrate how the foreign shocks propagate upstream or downstream. In doing so, we also contrast these outcomes with those from an economy where we assume that the network is fixed and does not change in response to the shock. Then, we consider an aggregate shock where all firms face a common increase in the costs of exporting. We illustrate how aggregate real income responds to this aggregate shock and demonstrate the role of endogenous networks by contrasting the responses to those when the network is fixed. Whenever we compute the fixed network counterfactual outcomes, we start from an economy that looks identical to the initial endogenous network economy in terms of firm-to-firm sales, firm-level labor costs, sales to final demand, and export and import participation. The system of equations to conduct the fixed network counterfactuals is provided in Appendix A.6.2.

As a first exercise, we focus on a 20 percent increase in the costs of exporting faced by a subset of firms. To illustrate how firm-level variables respond, we conduct the following simulation exercise. We randomly draw 100 direct exporters from our estimation sample of firms and increase the costs of exporting faced by these selected firms by 20 percent. We then solve for the equilibrium and calculate the average changes in firm-level outcomes for the 100 exporters as well as their direct suppliers, their suppliers, and so on. Most of these direct exporters have many suppliers, which may have heterogeneous exposure to the increase in costs of exporting. Instead of looking at the spillover effects on all of these suppliers, for each direct exporter, we focus on one key supplier that has the largest share of its revenue attributed to the direct exporter. Next, given the set of key suppliers, we look at the average changes in outcomes for the key suppliers upstream of these key suppliers, and so on. We repeat the above simulation 100 times (each time drawing 100 different exporters) and calculate the average across these 100 simulations.

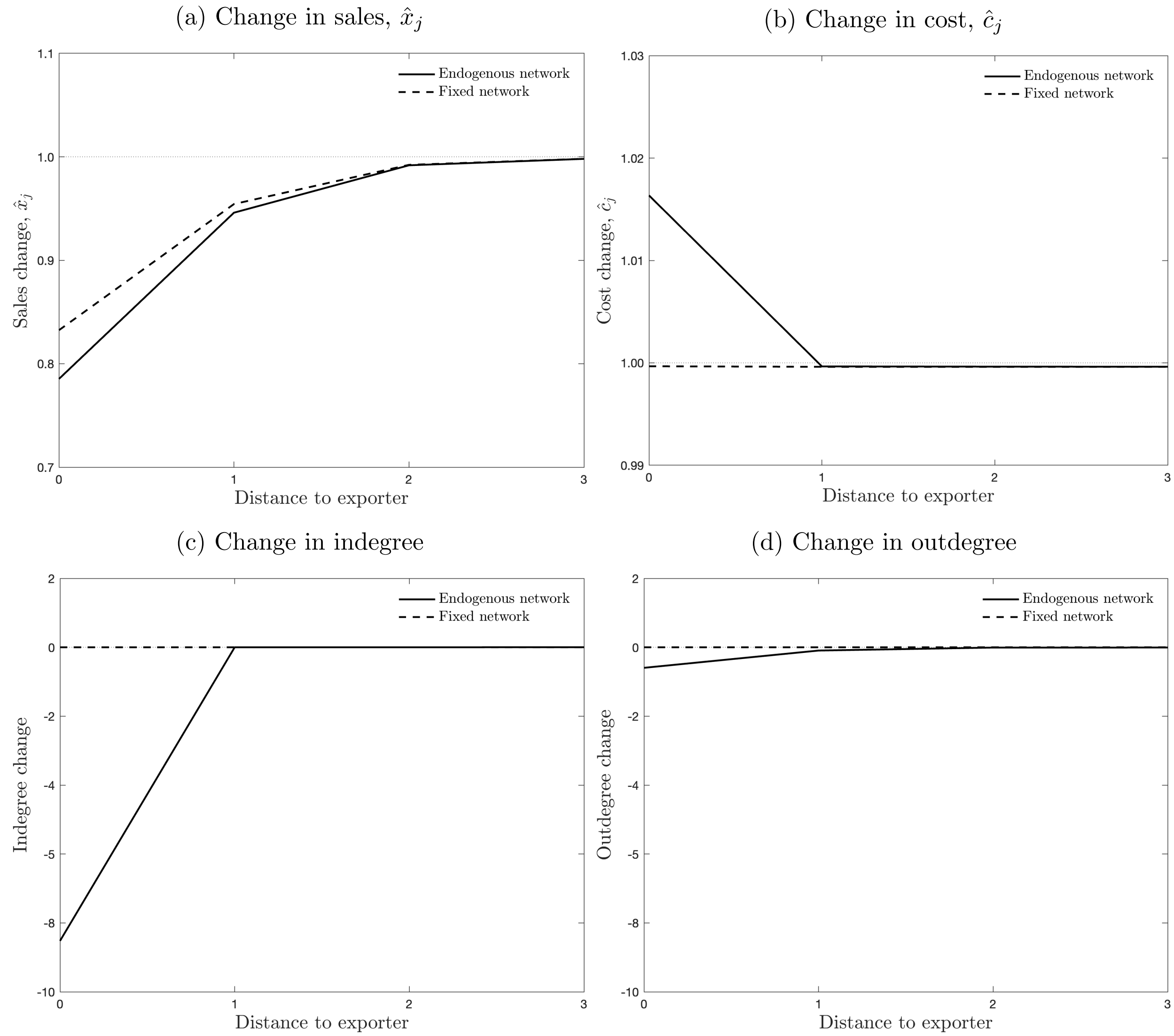
Figure 7 presents how firm-level outcomes change in response to the change in foreign demand for each group of firms that are defined according to the distance to the exporters who received the shock. The figure also displays the responses of the outcome variables when one imposes fixed networks. We display in Panel (a) how the change in foreign demand affects the sales of the direct exporters and propagates further upstream. Intuitively, the direct exporters experience the largest reduction in sales, followed by their suppliers, and then their suppliers, and so on. The panel also shows that on average, the direct exporters experience larger reductions in sales when firm-to-firm linkages are allowed to adjust in response to the shock. This difference in the magnitude of sales reduction among exporters is coming from the larger cost increases exporters face under endogenous networks (Panel (b)). To see this, note that the increase in the costs of exporting translates to lower profits for the direct exporters. The reduction in profits results in direct exporters not being able to pay for the fixed costs to link with their suppliers, leading to a fewer number of suppliers (i.e., lower indegree in Panel (c)) and larger cost increases. These larger cost increases also make the exporters less attractive as suppliers, resulting in a fewer number of firms that purchase from these exporters (i.e., lower outdegree in Panel (d)).

It is worth highlighting that the differences in these outcome variables between endogenous and fixed networks are quantitatively small once one focuses on the suppliers of the exporters, their suppliers, and so on. Recall that by assumption, buyers have full bargaining power in firm-to-firm trade and firms do not generate profits from their sales to other firms. This means that except for the general equilibrium effects, there are no direct effects on the profits of firms upstream to the exporters. Since profits do not change, their sourcing strategies and costs are not affected by demand shocks to their buyers. Therefore, in response to idiosyncratic foreign demand shocks, the main effects of allowing for endogenous network formation originate from the changes in the sourcing strategies of the exporters.

As the second exercise, we now focus on a 20 percent increase in the import prices, p_{Fj}, that a subset of importers face. Figure 8 shows how firm-level outcomes change in response to this 20 percent increase in import prices for a subset of firms. Similar to the previous exercise, firms are classified by their distances to the importers who received the import shock. We observe that the import shock propagates downstream from the direct importers: the direct importers face the largest reduction in sales, followed by their buyers, and then their buyers, and so on (Panel (a)). Moreover, when firm-to-firm linkages are allowed to adjust in response to the shock, the direct importers face a larger reduction in sales. This can be explained by the larger average increase in their costs (Panel (b)) since some firms cease to import in response to the higher import price. Aside from the general equilibrium effects, the increase in the import price does not affect the importers' decisions to source from other suppliers (i.e., unchanged indegree in Panel (c)). As explained in Section 3.1, the independent sourcing decisions across suppliers is coming from the fact that the elasticity of substitution in the production function and the demand elasticity are identical to each other. This makes the substitution and scale effects associated with the import-price cost shock cancel out each other. However, the increase in cost makes the importers less attractive as suppliers, and this shock propagates downstream to their buyers who are also facing larger costs. This results in a fewer number of firms that source from these importers as well as from their downstream buyers (i.e., lower outdegree in Panel (d)).

We now turn to the aggregate responses to the changes in the costs of exporting and illustrate how aggregate real income, $\frac{E}{P}$, responds differently under endogenous networks compared to the economy with fixed networks. By Lerner's symmetry, this aggregate foreign demand shock is

Figure 7: Transmission of a 20 percent increase in export costs along the production network

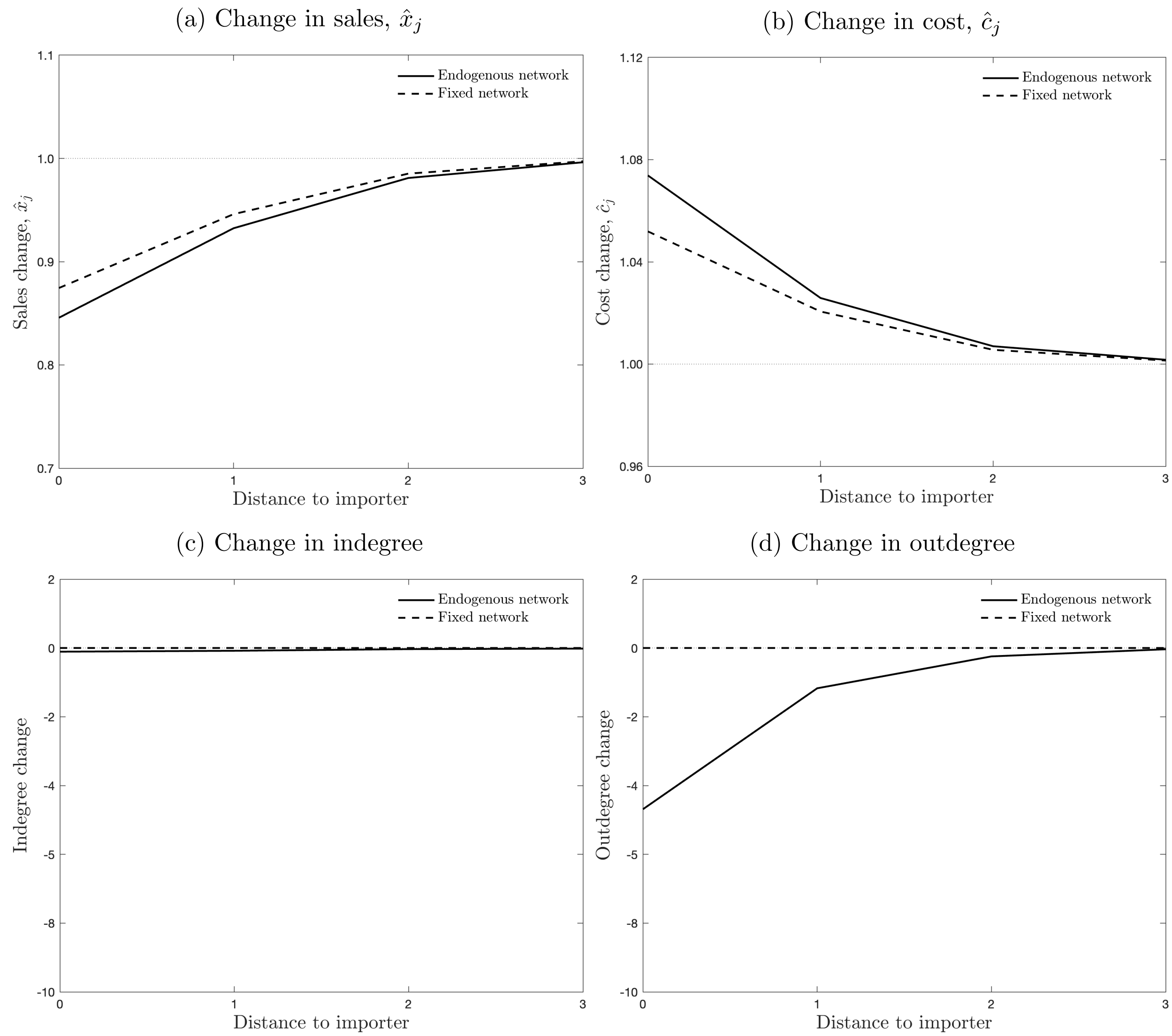


Notes: The figure shows the transmission of a 20 percent increase in export costs along the production network. First, we randomly draw 100 direct exporters from our estimation sample of firms and increase the export costs, τ_{jF}, faced by these selected firms by 20 percent. Then, we calculate the average changes in outcomes for these exporters. For each exporter, we select the key supplier that has the largest revenue share selling to the direct exporter, and we calculate the average changes in outcomes for those selected key suppliers. Next, given the set of key suppliers, we compute the average changes in outcomes for the key suppliers of these key suppliers and so on (up to three links). We independently conduct the above simulation 100 times (each time drawing 100 different exporters) and calculate the average across these 100 simulations. We compare changes in outcomes when the network is endogenous (indicated by solid lines) with those when the network is fixed (indicated by dashed lines).

equivalent to considering an aggregate change in the import price.[22] This implies that the aggregate foreign demand shock also affects some firms just as a foreign supply shock does, bringing into action both adjustments to demand shocks and adjustments to supply shocks, as characterized by the discussion above.

[22] In this case, a uniform change in the costs of exporting, $\hat{\tau}_{jF} = \hat{\tau}\ \forall j \in \Omega$, is equivalent to a uniform change in the import prices of the same percentage, $\hat{p}_{Fj} = \hat{\tau}\ \forall j \in \Omega$.

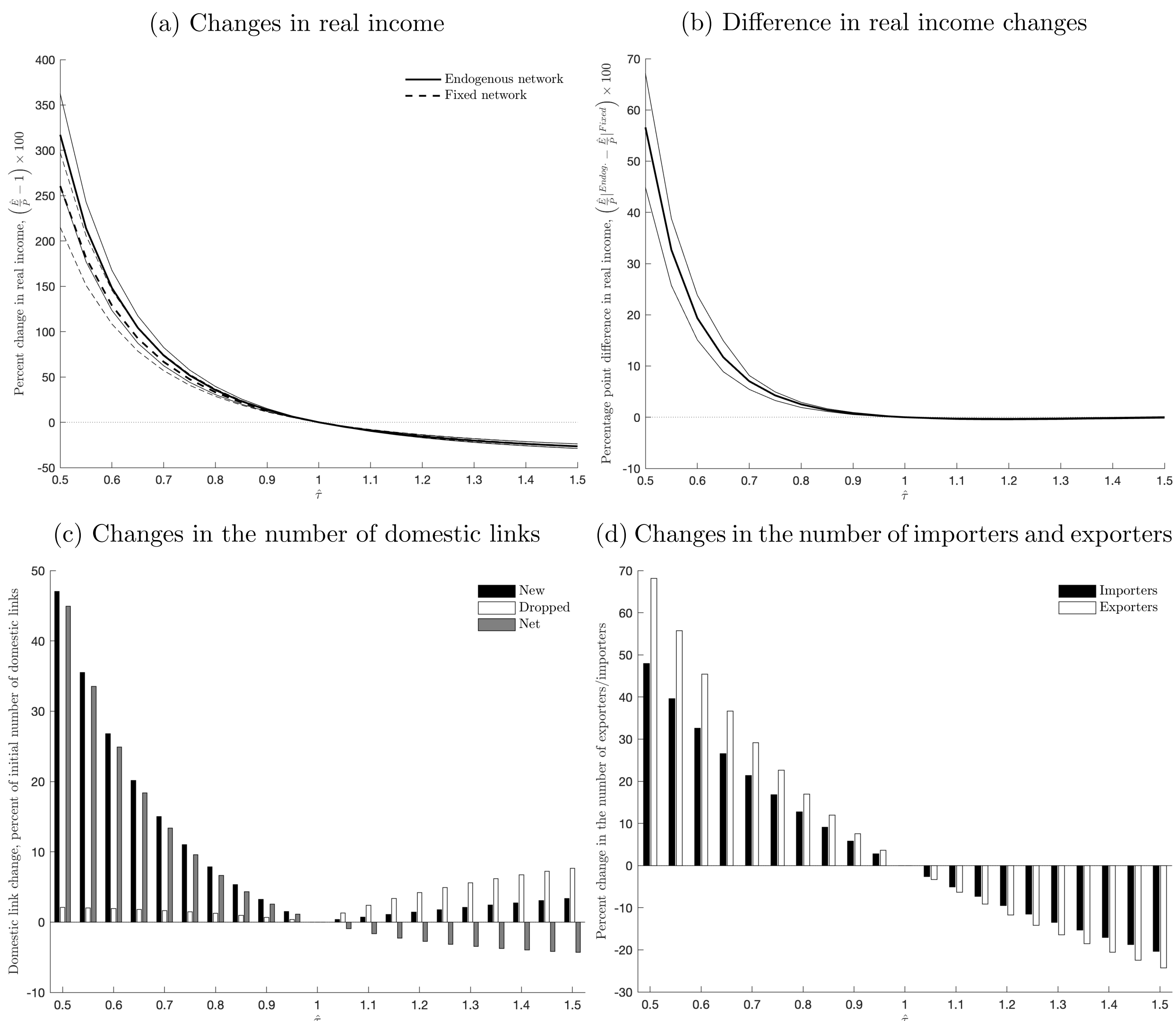
Figure 8: Transmission of a 20 percent increase in import prices along the production network



Notes: The figure shows the transmission of a 20 percent increase in import prices along the production network. First, we randomly draw 100 direct importers from our estimation sample of firms and increase the import prices, p_{Fj}, faced by these selected firms by 20 percent. Then, we calculate the average changes in outcomes for these importers. For each importer, we select the key buyer that has the largest cost share buying from the direct importer, and we calculate the average changes in outcomes for those selected key buyers. Next, given the set of key buyers, we compute the average changes in outcomes for the key buyers of these key buyers and so on (up to three links). We independently conduct the above simulation 100 times (each time drawing 100 different importers) and calculate the average across these 100 simulations. We compare changes in outcomes when the network is endogenous (indicated by solid lines) with those when the network is fixed (indicated by dashed lines).

In Panel (a) of Figure 9, we plot the real income responses to changes in export costs of different magnitudes, together with their 90 percent confidence intervals. The confidence interval is computed by simulating the counterfactual economy 40 times using the same set of estimated parameters but each with different realized draws of firms' technology parameters and fixed costs. Simulating the economy several times for different realized draws is important since for a given set of technology draws the endogenous network and fixed network response tend to diverge discontinuously at a certain threshold, with the threshold being sensitive to the particular set of technology draws. The panel shows a nonlinear relationship between the change in export costs and the change in real income. This asymmetry in the magnitude of real income changes between negative foreign demand shocks (when $\hat{\tau}_{jF} > 1$) and positive foreign demand shocks (when $\hat{\tau}_{jF} < 1$) is not surprising, as real income would rise to infinity as the export costs approach zero ($\hat{\tau}_{jF} \to 0$) but would converge to a finite number as the economy moves to autarky ($\hat{\tau}_{jF} \to \infty$).

Figure 9: Aggregate responses to changes in export costs



Notes: The figure reports the aggregate responses to changes in the costs of exporting, $\hat{\tau}_{jF} = \hat{\tau}\ \forall j \in \Omega$, from different simulations of the model given estimated parameters. Panel (a) plots the change in real income under an endogenous network and a fixed network in response to different levels of changes in the costs of exporting indicated on the horizontal axis. Panel (b) plots the difference in the real income change between the endogenous and fixed network models. The 90 percent confidence intervals are shown as the thin solid or dashed lines. Panel (c) plots the number of new domestic links, the number of dropped domestic links, and the net change in the number of domestic links (in percent, relative to the initial number of links). Panel (d) plots the changes in the number of importers and exporters (in percent, relative to the initial number of importers and exporters).

In Panel (b), we plot the percentage point differences in the real income changes depicted in Panel (a), together with the 90 percent confidence intervals. The panel reveals asymmetric effects of endogenous networks between positive and negative foreign demand shocks. For positive foreign demand shocks, allowing for endogenous networks magnifies the increase in aggregate real income but has a quantitatively negligible effect upon negative foreign demand shocks. This asymmetry can also be seen in Panel (c), where we plot the percent changes in the number of new domestic links, the number of dropped domestic links, as well as the net change in the number of domestic links relative to the baseline ($\hat{\tau}_{jF} = 1$). The number of domestic links that are newly formed upon positive demand shocks is much larger than the number of domestic links that are dropped upon negative demand shocks. One explanation is that when firms consider adding new suppliers in response to a positive demand shock, they face a large set of potential suppliers. In contrast, when firms consider dropping existing suppliers in response to a negative demand shock, they typically only choose from a small number of suppliers (Table 4). While Panel (c) shows that the number of net changes in domestic links changes non-linearly and is rather asymmetric between positive and negative shocks (correlated with what we see in Panel (b)), Panel (d) shows that the number of exporters and importers change rather linearly. This suggests that the non-linear effects on real income comes from the adjustment in the domestic firm-to-firm network, not from selection into importing and exporting.

Panel (b) also shows that the change in real income, $\hat{E}/\hat{P}$, is almost always larger under endogenous networks than under fixed networks. This is in line with the prediction under an efficient economy, where adding flexibility to the network structure would amplify the aggregate effect of a positive shock and dampen the aggregate effect of a negative shock. In other words, in an efficient economy, the difference in the aggregate real income change depicted in Panel (b) would always be positive. However, the endogenous network economy we analyze is not necessarily efficient since firms do not internalize the impact their sourcing decisions have on other firms in the economy, which explains why, for some changes in trade costs, the difference in the real income changes under endogenous networks and the real income change under fixed networks is slightly negative.

Our findings indicate that while trade shocks can induce sizable churn in domestic firm-to-firm linkages and they lead to changes in the aggregate real income, their quantitative implications, however, modest. Even when the costs of exporting increase by 50 percent, despite a four percent decrease in the number of domestic firm-to-firm linkages, the differences in the real income responses between the fixed and endogenous network models are negligible. Similarly, upon a decline in the costs of exporting by 5 percent, the number of domestic linkages increases by 1 percent, but the changes in real income are very similar between endogenous and fixed networks (6.1 percent versus 5.8 percent). Considering a very large decline of the costs of exporting by 20 percent (equivalent to a 144 percent increase in the foreign demand shifter D_F), while the number of domestic firm-to-firm linkages increases by around 7 percent, the resulting change in the real income under endogenous networks is still only moderately larger than that under exogenous networks (36 percent versus 33 percent).

The rather modest differences between the fixed and endogenous network models' real income changes can partially be explained by the fixed network economy already allowing for substitution between firms. The elasticity of substitution governs how easily firms and households can substitute across suppliers. To further examine the importance of the elasticity of substitution, we follow the procedure described in Section 4.3 and recalibrate the model for a low elasticity of substitution ($\sigma = 2$) and a high elasticity of substitution ($\sigma = 6$). Note that in terms of real income, the effect of a given change in export costs, $\hat{\tau}_{jF} = \hat{\tau}\ \forall j \in \Omega$, is equivalent to that of a change in the foreign demand shifter by $\hat{D}_F = \hat{\tau}^{-\sigma}$. This implies for a given change in export costs, $\hat{\tau}$, different values of elasticity of substitution imply different changes in the foreign demand shifter. Therefore, in Appendix C.2, we consider outcomes under different values of elasticity of substitution for a given change in export costs scaled by $-\sigma$ (i.e., $\hat{\tau}^{-\sigma}$). We find that lower elasticity of substitution between firms in the production and utility functions tends to generate larger adjustments in the extensive margin of firm-to-firm linkages. The magnitude in the adjustments in the extensive margin is also positively correlated with the difference in the real income changes between the endogenous and fixed network models.

5 Conclusion

A fundamental question in economics is what determines the shape of the firm-to-firm production network and, relatedly, how does the production network react to supply or demand shocks? This paper presents a tractable model of endogenous production networks with fixed costs of link formation. We establish results for the existence and uniqueness of the closed economy equilibrium. The conditions for equilibrium uniqueness are strong: all feasible networks must be acyclic, and the buyer initiates the network formation while having full bargaining power in the transaction with its supplier. We provide examples of multiple equilibria when deviating from these conditions. We show examples of multiple equilibria when the supplier pays the fixed costs of link formation while charging positive markups on firm-to-firm transactions or when the buyer pays the fixed cost of link formation but the network contains cycles. Our empirical results suggest that an acyclic production network model can approximate the observed firm-to-firm production network in several important dimensions. The quantitative analysis using a model where buyers pay the fixed costs and all feasible networks are acyclic suggests a moderate role for the endogeneity of domestic firm-to-firm linkages in shaping the aggregate response to trade shocks.

We see several avenues for future research. First, the tractability of our model of endogenous production networks allows one to apply the model to a range of questions in other contexts (e.g., studying the effects of domestic taxes and subsidies). Second, we think it would be valuable to incorporate positive markups on firm-to-firm transactions, as some of the prior research has done (e.g., Lim 2018, Dhyne et al. 2022b, and Bernard et al. 2022). Since our examples show that such markups on firm-to-firm transactions can lead to multiple equilibria, a new result on equilibrium uniqueness would have to be established or methods used that can allow for multiple equilibria in counterfactual analysis. Third, we think it would be valuable to more closely target the model to evidence from firm-level responses to demand or supply shocks. This should enhance the credibility of the counterfactual analysis.

References

Acemoglu, D. and P. D. Azar (2020): "Endogenous production networks," *Econometrica*, 88, 33–82.

Acemoglu, D. and A. Tahbaz-Salehi (2020): "Firms, Failures, and Fluctuations: The Macroeconomics of Supply Chain Disruptions," Working Paper 27565, National Bureau of Economic Research.

Alviarez, V., M. Fioretti, A. K. Kikkawa, and M. Morlacco (2022): "Two-sided Market Power in Firm-to-Firm Trade," Tech. rep., working paper.

Antras, P., T. C. Fort, and F. Tintelnot (2017): "The margins of global sourcing: Theory and evidence from us firms," *American Economic Review*, 107, 2514–64.

Antràs, P., T. C. Fort, and F. Tintelnot (2017): "The Margins of Global Sourcing: Theory and Evidence from US Firms," *American Economic Review*, 107, 2514–64.

Antràs, P. and E. Helpman (2004): "Global Sourcing," *Journal of Political Economy*, 112, 552–580.

Arkolakis, C., F. Huneeus, and Y. Miyauchi (2023): "Spatial Production Networks," *Unpublished, Yale University.*

Atalay, E., A. Hortacsu, J. Roberts, and C. Syverson (2011): "Network structure of production," *Proceedings of the National Academy of Sciences*, 108, 5199–5202.

Baqaee, D. and E. Farhi (2021): "Networks, Barriers, and Trade," Working Paper 26108, National Bureau of Economic Research.

Baqaee, D. R. and E. Farhi (2019a): "The Macroeconomic Impact of Microeconomic Shocks: Beyond Hulten's Theorem," *Econometrica*, 87, 1155–1203.

——— (2019b): "Productivity and Misallocation in General Equilibrium*," *The Quarterly Journal of Economics*, 135, 105–163.

Bernard, A. B., E. Dhyne, G. Magerman, K. Manova, and A. Moxnes (2022): "The origins of firm heterogeneity: A production network approach," *Journal of Political Economy*, 130, 1765–1804.

Bernard, A. B. and A. Moxnes (2018): "Networks and trade," *Annual Review of Economics*, 10, 65–85.

Black, P. E. (1999): "Algorithms and Theory of Computation Handbook," .

——— (2005): "Greedy Algorithm," *Dictionary of Algorithms and Data Structures*, 2.

Boehm, J. and E. Oberfield (2020): "Misallocation in the Market for Inputs: Enforcement and the Organization of Production*," *The Quarterly Journal of Economics*, 135, 2007–2058.

Carvalho, V. M. and A. Tahbaz-Salehi (2019): "Production networks: A primer," *Annual Review of Economics*, 11, 635–663.

Chaney, T. (2014): "The Network Structure of International Trade," *American Economic Review*, 104, 3600–3634.

Coleman, T. and A. Wirth (2010): "Ranking tournaments: Local search and a new algorithm," *Journal of Experimental Algorithmics (JEA)*, 14, 2–6.

de Loecker, J., C. Fuss, and J. Van Biesebroeck (2014): "International competition and firm performance: Evidence from Belgium," Tech. rep., NBB Working Paper.

Dekle, R., J. Eaton, and S. Kortum (2007): "Unbalanced Trade," *American Economic Review: Papers & Proceedings*, 97, 351–355.

Demir, B., A. C. Fieler, D. Xu, and K. K. Yang (2023): "O-ring production networks," *Journal of Political Economy.*

Dhyne, E., A. K. Kikkawa, T. Komatsu, M. Mogstad, and F. Tintelnot (2022a): "Foreign Demand Shocks to Production Networks: Firm Responses and Worker Impacts," Tech. rep., National Bureau of Economic Research.

Dhyne, E., A. K. Kikkawa, and G. Magerman (2022b): "Imperfect Competition in Firm-to-Firm Trade," *Journal of the European Economic Association*, jvac013.

Dhyne, E., A. K. Kikkawa, M. Mogstad, and F. Tintelnot (2021): "Trade and domestic production networks," *The Review of Economic Studies*, 88, 643–668.

——— (2023): "Measuring the Share of Imports in Consumption," *American Economic Association Papers and Proceedings.*

Dhyne, E., G. Magerman, and S. Rubínová (2015): "The Belgian production network 2002-2012," Tech. rep., NBB Working Paper.

Dixit, A. K. and J. E. Stiglitz (1977): "Monopolistic Competition and Optimum Product Diversity," *The American Economic Review*, 67, 297–308.

Eades, P., X. Lin, and W. F. Smyth (1993): "A fast and effective heuristic for the feedback arc set problem," *Information Processing Letters*, 47, 319–323.

Eaton, J., S. Kortum, and F. Kramarz (2011): "An Anatomy of International Trade: Evidence From French Firms," *Econometrica*, 79, 1453–1498.

——— (2022): "Firm-to-Firm Trade: Imports, Exports, and the Labor Market," Tech. rep., working paper.

Hulten, C. R. (1978): "Growth Accounting with Intermediate Inputs," *The Review of Economic Studies*, 45, 511–518.

Huneeus, F. (2018): "Production network dynamics and the propagation of shocks," Tech. rep., working paper.

Huneeus, F., K. Kroft, and K. Lim (2021): "Earnings Inequality in Production Networks," Working Paper 28424, National Bureau of Economic Research.

Jackson, M. and B. Rogers (2007): "Meeting Strangers and Friends of Friends: How Random Are Social Networks ?" *American Economic Review.*

Jia, P. (2008): "What happens when Wal-Mart comes to town: An empirical analysis of the discount retailing industry," *Econometrica*, 76, 1263–1316.

Lim, K. (2018): "Endogenous Production Networks and the Business Cycle," Working paper.

Liu, E. (2019): "Industrial Policies in Production Networks*," *The Quarterly Journal of Economics*, 134, 1883–1948.

Melitz, M. J. (2003): "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity," *Econometrica*, 71, 1695–1725.

Miyauchi, Y. (2021): "Matching and Agglomeration: Theory and Evidence from Japanese Firm-to-Firm Trade," Tech. rep., working paper.

Oberfield, E. (2018): "A Theory of Input–Output Architecture," *Econometrica*, 86, 559–589.

Oberfield, E. and D. Raval (2021): "Micro Data and Macro Technology," *Econometrica*, 89, 703–732.

Panigrahi, P. (2021): "Endogenous Spatial Production Networks: Quantitative Implications for Trade and Productivity," Working paper.

Plemmons, R. J. (1977): "M-matrix characterizations. I—nonsingular M-matrices," *Linear Algebra and its applications*, 18, 175–188.

Simpson, M., V. Srinivasan, and A. Thomo (2016): "Efficient computation of feedback arc set at web-scale," *Proceedings of the VLDB Endowment*, 10, 133–144.

Stiglitz, J. E. (1970): "Non-substitution theorems with durable capital goods," *The Review of Economic Studies*, 37, 543–553.

Taschereau-Dumouchel, M. (2020): "Cascades and fluctuations in an economy with an endogenous production network," *Available at SSRN 3115854*.

Train, K. E. (2009): *Discrete choice methods with simulation*, Cambridge university press.

A Model Appendix

A.1 Existence and Uniqueness of Fixed Network Equilibrium

In this section, we discuss sufficient conditions under which there exists a unique closed economy equilibrium with fixed production networks. First, we characterize the existence and uniqueness of the vector of marginal costs. Equation (4) yields the following linear system in $c_j^{1-\sigma}$:

$$c_j^{1-\sigma} = \phi_j^{\sigma-1}\left(\sum_{k\in\Omega}\mathbb{I}\{k \in Z_j\}\alpha_{kj}^{\sigma-1}\mu_{kj}^{1-\sigma}c_k^{1-\sigma} + \alpha_{Lj}^{\sigma-1}w^{1-\sigma}\right).$$

In matrix notation, the equation above can be expressed as

$$\tilde{\mathbf{c}} = \mathbf{\Gamma}'\tilde{\mathbf{c}} + \tilde{\mathbf{\Phi}}\tilde{\boldsymbol{\alpha}}_L w^{1-\sigma}, \tag{A1}$$

where $\tilde{\mathbf{c}}$ is a vector of $c_j^{1-\sigma}$, $\mathbf{\Gamma}$ is a matrix with element (k,j), $k \neq j$, being $\mathbb{I}\{k \in Z_j\}\alpha_{kj}^{\sigma-1}\mu_{kj}^{1-\sigma}\phi_j^{\sigma-1} \geq 0$ and element (j,j) being 0 (since a firm does not source from itself), $\tilde{\mathbf{\Phi}} = \text{diag}(\phi_1^{\sigma-1}, \ldots, \phi_N^{\sigma-1})$ with $\phi_j > 0$ for all $j \in \Omega$, and $\tilde{\boldsymbol{\alpha}}_L$ is a vector of $\alpha_{Lj}^{\sigma-1} > 0$. The following lemma provides the sufficient condition under which the solution to equation (A1) is well defined and unique.[23]

Lemma 3. *Given a normalization for the wage level, w, there exists a unique $\tilde{\mathbf{c}}^* > 0$ that satisfies equation* (A1) *if the spectral radius of $\mathbf{\Gamma}'$ is less than one.*

Proof. Re-arrange equation (A1) and obtain:

$$\left(\mathbf{I} - \mathbf{\Gamma}'\right)\tilde{\mathbf{c}} = \tilde{\mathbf{\Phi}}\tilde{\boldsymbol{\alpha}}_L w^{1-\sigma},$$

where $\mathbf{\Gamma}'$ is a non-negative matrix, and $\tilde{\mathbf{\Phi}}\tilde{\boldsymbol{\alpha}}_L w^{1-\sigma}$ is a positive vector. Suppose the spectral radius of $\mathbf{\Gamma}'$ is less than one. It follows that $\mathbf{I}-\mathbf{\Gamma}'$ is a non-singular M-matrix, which ensures that $(\mathbf{I}-\mathbf{\Gamma}')^{-1}$ exists and is non-negative with at least one positive element in each row.[24] Therefore, the solution $\tilde{\mathbf{c}}^*$ given by

$$\tilde{\mathbf{c}}^* = \left(\mathbf{I} - \mathbf{\Gamma}'\right)^{-1}\tilde{\mathbf{\Phi}}\tilde{\boldsymbol{\alpha}}_L w^{1-\sigma},$$

is well defined and unique. □

We provide two sufficient conditions such that the spectral radius of $\mathbf{\Gamma}'$ is less than one. First, when the production network is acyclic, $\mathbf{\Gamma}'$ is strictly lower triangular, which implies that the spectral radius is zero. Second, when $\mathbf{I} - \mathbf{\Gamma}'$ is strictly diagonally dominant (i.e., $\sum_{k\neq j}\mathbb{I}\{k \in Z_j\}\left(\frac{\phi_j\alpha_{kj}}{\mu_{kj}}\right)^{\sigma-1} < 1$ for all $j \in \Omega$), the spectral radius of $\mathbf{\Gamma}'$ is less than one.[25]

[23] Stiglitz (1970) shows in Theorem 1 that there exists at most one positive cost vector. We provide the condition under which the positive cost vector is guaranteed to exist.

[24] Plemmons (1977) shows in Theorem 2 that the inverse of any non-singular M-matrix is nonnegative.

[25] This is implied by the Gershgorin circle theorem. First, note that the diagonal elements of $\mathbf{\Gamma}'$ are zero. Define

Next, consider the existence and uniqueness of the aggregate expenditure, E. Suppose that the cost vector exists and is unique, then the price index, P, and the sourcing capabilities, Θ_j, are determined. Then, if the vectors of sales and profits are well defined and unique, so is E. It is less clear, however, that there exists a unique sales vector because equation (8) is a system of nonlinear equations in x_j due to endogenous average markup $\bar{\mu}_j$.

We provide two different assumptions on the markups which allow us to simplify the analysis. First, suppose that the markups of all firm-to-firm transactions are one (i.e., $\mu_{jk} = 1$ for all $k \in B_j$ and $j \in \Omega$). In this case, firms do not profit from selling to other firms, and equation (11) simplifies to

$$E = wL + \sum_{j \in \Omega} \pi_{jH} = wL + \sum_{j \in \Omega} \left(1 - \frac{1}{\mu_{jH}}\right) \beta_{jH}^{\sigma-1} \mu_{jH}^{1-\sigma} \phi_j^{\sigma-1} \Theta_j \frac{E}{P^{1-\sigma}}$$

where π_{jH} is firm j's profits from selling to final consumers. Aggregate expenditure E can be solved explicitly:

$$E = \left(1 - \sum_{j \in \Omega} \left(1 - \frac{1}{\mu_{jH}}\right) \frac{\beta_{jH}^{\sigma-1} \mu_{jH}^{1-\sigma} \phi_j^{\sigma-1} \Theta_j}{\sum_{k \in \Omega} \beta_{kH}^{\sigma-1} \mu_{kH}^{1-\sigma} \phi_k^{\sigma-1} \Theta_k}\right)^{-1} wL.$$

Second, suppose that each firm charges a markup that does not vary across the identity of its buyers (i.e., $\mu_{jH} = \mu_{jk}$ for all $k \in B_j$ and $j \in \Omega$). Note that this allows the markup to vary across suppliers. In addition, assume that all markups are no less than one. In this case, we show in Proposition 2 below that the fixed network equilibrium is unique when the assumptions in Lemma 3 are satisfied.

Proposition 2. *Given a normalization for the wage level, w, there exists a unique equilibrium in Definition 1 when the spectral radius of $\mathbf{\Gamma}'$ is less than one and each firm charges a markup that does not vary across the identity of its buyers and is no less than one.*

Proof. First, Lemma 3 ensures that the cost vector is unique given the assumptions stated in the proposition. This implies that the vector of sourcing capabilities, Θ_j, and the price index, P, are uniquely determined.

Next, we show that the vector of sales, x_j, is uniquely determined. Recall that the vector of

γ_j as the sum of the absolute values of the off-diagonal elements in the j-th row of $\mathbf{\Gamma}'$:

$$\gamma_j = \sum_{k \neq j} \left| \mathbb{I}\{k \in Z_j\} \left(\frac{\phi_j \alpha_{kj}}{\mu_{kj}}\right)^{\sigma-1} \right|.$$

The theorem states that each eigenvalue of $\mathbf{\Gamma}'$ lies within at least one of the closed circles centered at 0 (the j-th diagonal element of $\mathbf{\Gamma}'$) with radius γ_j. Strict diagonal dominance of $\mathbf{I} - \mathbf{\Gamma}'$ implies that $\max_j \gamma_j < 1$. This means that the spectral radius of $\mathbf{\Gamma}'$ is less than one.

sales is characterized by equation (8):

$$x_j = \beta_{jH}^{\sigma-1}\mu_{jH}^{1-\sigma}\phi_j^{\sigma-1}\Theta_j\frac{E}{P^{1-\sigma}} + \sum_{k\in B_j}\alpha_{jk}^{\sigma-1}\mu_{jk}^{1-\sigma}\phi_j^{\sigma-1}\Theta_j\frac{x_k}{\Theta_k\bar{\mu}_k}.$$

Define the demand shifter for firm j as $G_j = \frac{x_j}{c_j^{1-\sigma}} = \frac{x_j}{\phi_j^{\sigma-1}\Theta_j}$ and the demand shifter for domestic households as $A = \frac{E}{P^{1-\sigma}}$. We can re-write equation (8) using the relative demand shifter, $\frac{G_j}{A}$:

$$\begin{aligned}
x_j &= \beta_{jH}^{\sigma-1}\mu_{jH}^{1-\sigma}\phi_j^{\sigma-1}\Theta_j A + \sum_{k\in B_j}\alpha_{jk}^{\sigma-1}\mu_{jk}^{1-\sigma}\phi_j^{\sigma-1}\Theta_j\frac{1}{\bar{\mu}_k}\frac{x_k}{\Theta_k}\\
\implies \frac{x_j}{\phi_j^{\sigma-1}\Theta_j A} &= \beta_{jH}^{\sigma-1}\mu_{jH}^{1-\sigma} + \sum_{k\in B_j}\alpha_{jk}^{\sigma-1}\mu_{jk}^{1-\sigma}\frac{1}{\bar{\mu}_k}\frac{x_k}{\Theta_k A}\\
\implies \frac{x_j}{\phi_j^{\sigma-1}\Theta_j A} &= \beta_{jH}^{\sigma-1}\mu_{jH}^{1-\sigma} + \sum_{k\in B_j}\alpha_{jk}^{\sigma-1}\mu_{jk}^{1-\sigma}\phi_k^{\sigma-1}\frac{1}{\bar{\mu}_k}\frac{x_k}{\phi_k^{\sigma-1}\Theta_k A}\\
\implies \frac{G_j}{A} &= \beta_{jH}^{\sigma-1}\mu_{jH}^{1-\sigma} + \sum_{k\in B_j}\alpha_{jk}^{\sigma-1}\mu_{jk}^{1-\sigma}\phi_k^{\sigma-1}\frac{1}{\bar{\mu}_k}\frac{G_k}{A}. \qquad \text{(A2)}
\end{aligned}$$

Note that when each firm charges a markup that does not vary across the identity of its buyers, $\mu_{jH} = \mu_{jk}$ for all k, which implies the average markup for firm j, $\bar{\mu}_j$, is exogenously given. Therefore, equation (A2) is a linear system that determines the vector of relative demand shifters, $\frac{G_j}{A}$.

We show that the vector of relative demand shifters, $\frac{G_j}{A}$, is uniquely determined by equation (A2) under the assumptions stated in the proposition. Note that equation (A2) can be represented in matrix notation as follows:

$$\tilde{\mathbf{G}} = \boldsymbol{\lambda} + \boldsymbol{\Gamma}\tilde{\boldsymbol{\mu}}^{-1}\tilde{\mathbf{G}},$$

where $\tilde{\mathbf{G}}$ is the vector of relative demand shifters, $\frac{G_j}{A}$, $\boldsymbol{\lambda}$ is the vector of $\beta_{jH}^{\sigma-1}\mu_{jH}^{1-\sigma}$, $\boldsymbol{\Gamma}$ follows the definitions in equation (A1), and $\tilde{\boldsymbol{\mu}} = \text{diag}(\bar{\mu}_1, \ldots, \bar{\mu}_N)$.

We note two important observations. First, matrices $\boldsymbol{\Gamma}'$ and $\boldsymbol{\Gamma}$ share a common set of eigenvalues. Denote the spectral radius of a matrix M as $\rho(M)$. Then, $\rho(\boldsymbol{\Gamma}) = \rho(\boldsymbol{\Gamma}') < 1$. Second, since $\boldsymbol{\Gamma}$ is element-wise non-negative and $\tilde{\boldsymbol{\mu}}^{-1}$ is a diagonal matrix with positive diagonal elements that are no greater than one, we have $\rho(\boldsymbol{\Gamma}\tilde{\boldsymbol{\mu}}^{-1}) < 1$.[26] This means the spectral radius of the coefficient matrix in equation (A2) is less than one. Following the proof of Lemma 3, there exists a unique

[26] To see this, note that the spectral radius of a matrix M, $\rho(M)$, is less than one if and only if $\lim_{k\to\infty} M^k = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix. In other words, for any $\epsilon > 0$, there exists an integer N_ϵ such that $||M^k||_{\max} < \epsilon$ for all $k > N_\epsilon$, where $||M||_{\max} = \max_{ij}|m_{ij}|$ is the max norm of the matrix M. Setting $M = \boldsymbol{\Gamma}$, $\rho(\boldsymbol{\Gamma}) < 1$ implies that for any $\epsilon > 0$, there exists $N_\epsilon > 0$ such that $||(\boldsymbol{\Gamma})^k||_{\max} < \epsilon$ for all $k > N_\epsilon$. Consider entry (i,j) of $\boldsymbol{\Gamma}$, denoted as γ_{ij}, then γ_{ij}/μ_j is entry (i,j) of $\boldsymbol{\Gamma}\tilde{\boldsymbol{\mu}}^{-1}$. For all i,j, we have $0 \leq \gamma_{ij}/\mu_j \leq \gamma_{ij}$ since $\gamma_{ij} \geq 0$ and $\mu_j \geq 1$. This implies $||\boldsymbol{\Gamma}\tilde{\boldsymbol{\mu}}^{-1}||_{\max} \leq ||\boldsymbol{\Gamma}||_{\max}$. Moreover, $||(\boldsymbol{\Gamma}\tilde{\boldsymbol{\mu}}^{-1})^k||_{\max} \leq ||(\boldsymbol{\Gamma})^k||_{\max}$ for all integers k. Then, for any $\epsilon > 0$, there exists an integer N_ϵ such that $||(\boldsymbol{\Gamma}\tilde{\boldsymbol{\mu}}^{-1})^k||_{\max} \leq ||(\boldsymbol{\Gamma})^k||_{\max} < \epsilon$ for all $k > N_\epsilon$. Therefore, $\lim_{k\to\infty}(\boldsymbol{\Gamma}\tilde{\boldsymbol{\mu}}^{-1})^k = \mathbf{0}$, which implies that $\rho(\boldsymbol{\Gamma}\tilde{\boldsymbol{\mu}}^{-1}) < 1$.

$\tilde{\mathbf{G}}^* > 0$ that satisfies equation (A2), where

$$\tilde{\mathbf{G}}^* = \left(\mathbf{I} - \mathbf{\Gamma}\tilde{\boldsymbol{\mu}}^{-1}\right)^{-1} \boldsymbol{\lambda}.$$

We have established that the vector of relative demand shifters, $\frac{G_j}{A}$, is uniquely determined. This implies that the level of firm-specific demand shifter, G_j, is unique up to a level of domestic household demand, A. Using the labor market clearing condition (equation (12)), one can pin down the level of A.

$$\begin{aligned} wL &= \sum_{j\in\Omega} \frac{1}{\bar{\mu}_j} s_{Lj} x_j = \sum_{j\in\Omega} \frac{1}{\bar{\mu}_j} s_{Lj} c_j^{1-\sigma} G_j = \sum_{j\in\Omega} \frac{1}{\bar{\mu}_j} s_{Lj} c_j^{1-\sigma} \frac{G_j}{A} A \\ \implies A &= wL \left(\sum_{j\in\Omega} \frac{1}{\bar{\mu}_j} s_{Lj} c_j^{1-\sigma} \frac{G_j}{A} \right)^{-1}. \end{aligned}$$

Given A, the firm-specific demand shifters, G_j, are pinned down. It is straightforward to show that the vector of sales as well as the remaining equilibrium variables are uniquely determined. □

A.2 Existence and Uniqueness of Equilibrium with Buyers Initiating Contacts

Proof of Lemma 1

Proof. From the definition of $\Theta_j(\cdot)$, $\Theta_j(Z_j') - \Theta_j(Z_j) = \sum_{k\in Z_j' \setminus Z_j} \alpha_{kj}^{\sigma-1} \mu_{kj}^{1-\sigma} c_k^{1-\sigma} \geq 0$ implies that $\Theta_j(Z_j') \geq \Theta_j(Z_j)$. Since $c_j(\cdot) = \frac{1}{\phi_j}\Theta_j(\cdot)^{\frac{1}{1-\sigma}}$, $c_j(Z_j') \leq c_j(Z_j)$. □

Proof of Lemma 2

Proof. We prove the lemma by induction. Consider firm $j > 1$. Suppose that for each potential supplier k, $Z_k(A) \subset Z_k(A')$ holds. Firm j includes supplier k in its sourcing strategy Z_j if the increase in variable profits (marginal benefit, MB) is no less than the fixed cost of link formation (marginal cost, MC):

$$MB = \left(1 - \frac{1}{\mu_{jH}}\right) \beta_{jH}^{\sigma-1} \mu_{jH}^{1-\sigma} \phi_j^{\sigma-1} \alpha_{kj}^{\sigma-1} \mu_{kj}^{1-\sigma} c_k(Z_k(A))^{1-\sigma} A \geq f_{kj} w = MC$$

For any $k \in Z_j(A)$, we want to show that $k \in Z_j(A')$. Note that since $Z_k(A) \subset Z_k(A')$ holds by assumption, we know from Lemma 1 that $c_k(Z_k(A')) \leq c_k(Z_k(A))$. This implies that the marginal benefit of including k as a supplier is non-decreasing, whereas the fixed cost does not change. Therefore, firm j still keeps supplier k in its sourcing strategy, and $k \in Z_j(A')$.

It remains to show that $Z_1(A) \subset Z_1(A')$ holds for firm 1. This is trivial because $Z_1(A) = Z_1(A') = \emptyset$, which concludes the proof. □

Proof of Proposition 1

Proof. Note that $A_1(A_0)$, obtained from the solution algorithm in Appendix A.6.1, is given by

$$A_1(A_0) = \frac{E(A_0)}{P(A_0)^{1-\sigma}} = \frac{wL + \sum_{j\in\Omega}\left(\left(1-\frac{1}{\mu_{jH}}\right)\beta_{jH}^{\sigma-1}\mu_{jH}^{1-\sigma}\phi_j^{\sigma-1}\Theta_j(Z_j(A_0))A_0 - \sum_{k\in Z_j(A_0)} f_{kj}w\right)}{\sum_{j\in\Omega}\beta_{jH}^{\sigma-1}\mu_{jH}^{1-\sigma}\phi_j^{\sigma-1}\Theta_j(Z_j(A_0))}.$$

Since $\mu_{jH} = \mu_H$ for all j, we can simplify the equation above:

$$A_1(A_0) - A_0 = -\frac{1}{\mu_H}A_0 + \frac{wL - \sum_{j\in\Omega}\sum_{k\in Z_j(A_0)} f_{kj}w}{\mu_H^{1-\sigma}\sum_{j\in\Omega}\beta_{jH}^{\sigma-1}\phi_j^{\sigma-1}\Theta_j(Z_j(A_0))}.$$

Notice that the first term directly depends on A_0, whereas the second term indirectly depends on A_0 through $Z_j(A_0)$. Suppose $A_0' > A_0$. Consider two cases:

If $Z_j(A_0') = Z_j(A_0)$ for all $j \in \Omega$ (i.e., the network structure does not change), then the second term on the right-hand side of equation (18) does not change. Therefore, the derivative of $A_1(A_0) - A_0$ with respect to A_0 is $-\frac{1}{\mu_H}$.

If $Z_j(A_0') \neq Z_j(A_0)$ for some $j \in \Omega$ (i.e., the sourcing strategy changes for some firms), then $A_1(A_0) - A_0$ changes discontinuously. To characterize the direction of the discontinuous change, we analyze how the second term on the right-hand side of equation (18) changes. From Lemma 2, we know that the sourcing strategies are weakly expanding for all firms and strictly expanding for some firms. This implies that the total fixed costs increase (smaller numerator). On the other hand, Lemma 1 shows that sourcing capabilities Θ weakly increase for all firms and strictly increase for some firms (greater denominator). As a result, the discontinuous change in $A_1(A_0) - A_0$ is negative.

Since both the continuous change and the discontinuous change (if the network structure changes) are negative, we conclude that $A_1(A_0) - A_0$ is decreasing in A_0. □

A.3 Logit Smoothing

In this section, we describe the method of logit smoothing. We assume that each firm type consists of a continuum of individual firms, denoted $\omega \in [0, 1]$. Each firm is infinitesimal and independently chooses whether to source from each potential supplier. Firms can only buy inputs from the entire firm type, which is the integral of all the individual firms. When evaluating the profitability of adding an potential supplier, each firm has private information about its idiosyncratic taste regarding whether or not to source from the supplier. Specifically, the infinitesimal firm ω of firm type j sources from firm type k if

$$v_{kj}(\omega) + \lambda\epsilon_{kj}^1(\omega) \geq \lambda\epsilon_{kj}^0(\omega),$$

where

$$v_{kj}(\omega) = \left(1 - \frac{1}{\mu_{jH}}\right)\beta_{jH}^{\sigma-1}\mu_{jH}^{1-\sigma}\phi_j^{\sigma-1}\alpha_{kj}^{\sigma-1}\mu_{kj}^{1-\sigma}\phi_k^{\sigma-1}\Theta_k A - f_{kj}w$$

are the net benefits for decision maker ω to add firm type k to firm type j's sourcing strategy, $\epsilon^1_{kj}(\omega)$ and $\epsilon^0_{kj}(\omega)$ are i.i.d. T1EV shocks, and λ is the logit smoothing parameter governing the dispersion of the shocks. Notice that when λ is close to 0, the infinitesimal firm ω includes firm type k in its set of suppliers as long as $v_{kj}(\omega) \geq 0$, collapsing to the case where each type has only one firm. The probability that the infinitesimal firm ω of firm type j sources from firm type k is

$$\mathsf{p}_{kj}(\omega) = \frac{\exp(v_{kj}(\omega)/\lambda)}{1 + \exp(v_{kj}(\omega)/\lambda)}.$$

As all infinitesimal firms collectively determine the overall decision of firm type j, we obtain the probability that firm type j sources from firm type k as

$$\mathsf{p}_{kj} = \int_0^1 \mathsf{p}_{kj}(\omega) d\omega.$$

With a slight abuse of notation, we express firm type j's *expected* sourcing capability as

$$\begin{aligned}\Theta_j &= \sum_{k \in \mathbf{Z}_j} \int_0^1 \mathsf{p}_{kj}(\omega) \alpha_{kj}^{\sigma-1} p_{kj}^{1-\sigma} d\omega + \alpha_{Lj}^{\sigma-1} w^{1-\sigma} \\ &= \sum_{k \in \mathbf{Z}_j} \mathsf{p}_{kj} \alpha_{kj}^{\sigma-1} p_{kj}^{1-\sigma} + \alpha_{Lj}^{\sigma-1} w^{1-\sigma}.\end{aligned}$$

The marginal cost of firm type j is given by $c_j = \frac{1}{\phi_j} \Theta_j^{\frac{1}{1-\sigma}}$. Firms' sourcing problems can still be solved sequentially given a guess for domestic household demand. The key difference is that downstream firms take the expected sourcing capabilities of upstream firms as given.

Now, firm type j's total sales are characterized by

$$x_j = \beta_{jH}^{\sigma-1} \mu_H^{1-\sigma} \phi_j^{\sigma-1} \Theta_j \frac{E}{P^{1-\sigma}} + \sum_{k \in \Omega} \mathbb{I}\{j \in \mathbf{Z}_k\} \mathsf{p}_{jk} \alpha_{jk}^{\sigma-1} \mu_{jk}^{1-\sigma} \phi_j^{\sigma-1} \Theta_j \frac{x_k}{\Theta_k \bar{\mu}_k},$$

and its variable profits are given by

$$\pi_j^{var} = \left(1 - \frac{1}{\mu_H}\right) x_{jH} = \left(1 - \frac{1}{\mu_H}\right) \beta_{jH}^{\sigma-1} \mu_H^{1-\sigma} \phi_j^{\sigma-1} \Theta_j \frac{E}{P^{1-\sigma}}.$$

We can then compute the gross profits of firm type j as

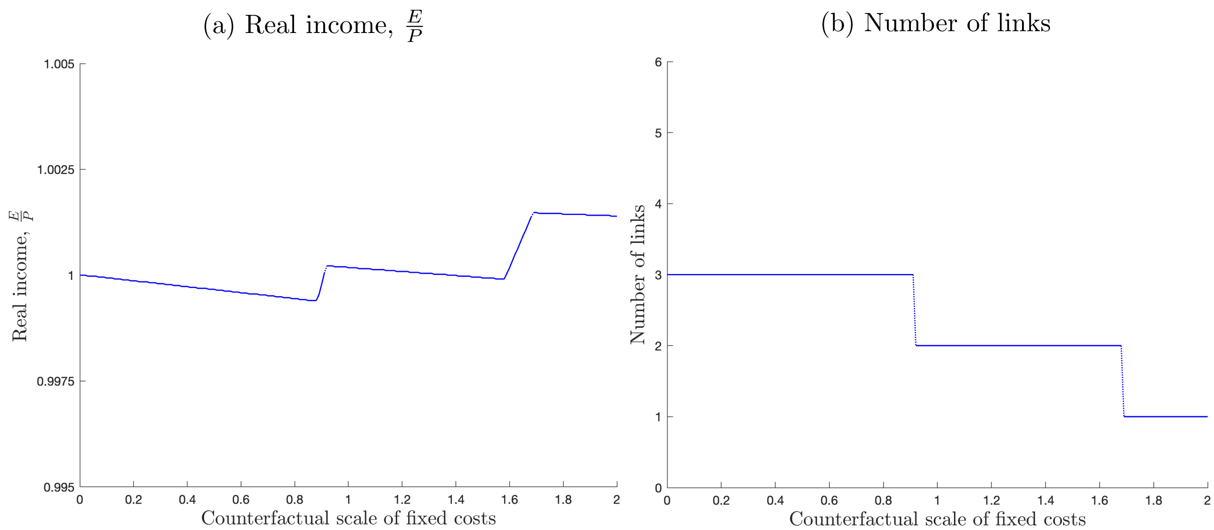
$$\pi_j = \pi_j^{var} - \sum_{k \in \mathbf{Z}_j} \mathsf{p}_{kj} f_{kj} w.$$

Since the probabilities of link formation $\{\mathsf{p}_{kj}\}$ continuously adjust in response to changes in domestic household demand, changes in sourcing capabilities, marginal costs, and implied domestic household demand are smooth.

A.4 Three-Firm Example: Additional Results

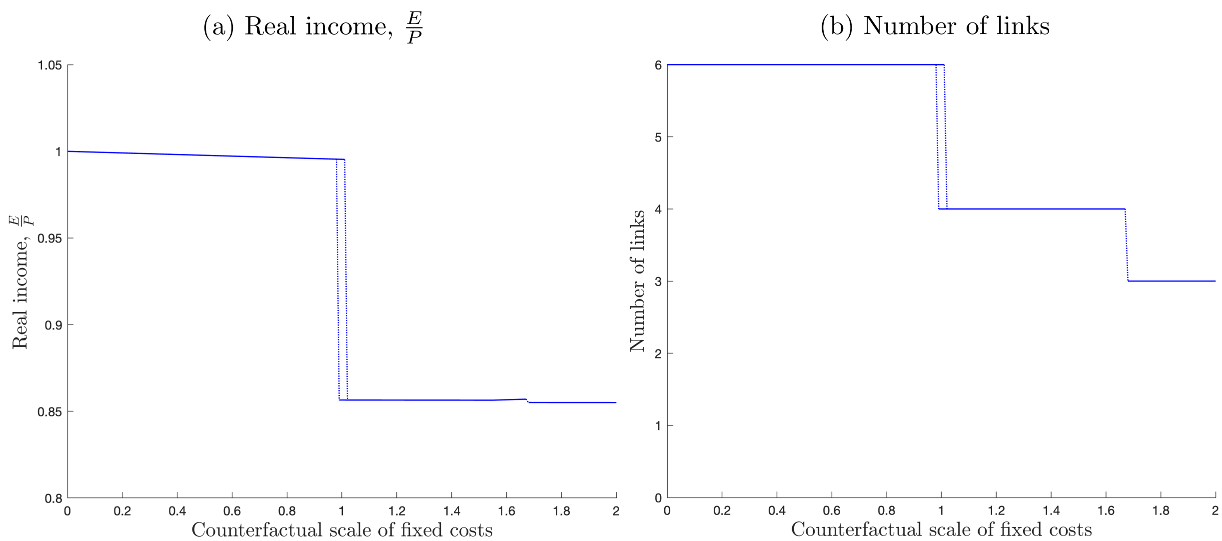
In this section, we provide additional results on the three-firm examples discussed in Section 3.2. Specifically, we consider counterfactual scales of fixed costs and analyze the changes in real income and aggregate number of links. Figures A1 to A4 show the real income (panel (a)) and the aggregate number of links (panel (b)) against different scales of fixed costs for four combinations of contact initiators and network structures. We illustrate that in response to the same change in fixed costs, the number of connections and real income may respond in different magnitudes depending on which equilibrium is realized.

Figure A1: Counterfactual scale of fixed costs: endogenous acyclic network with buyers initiating contacts



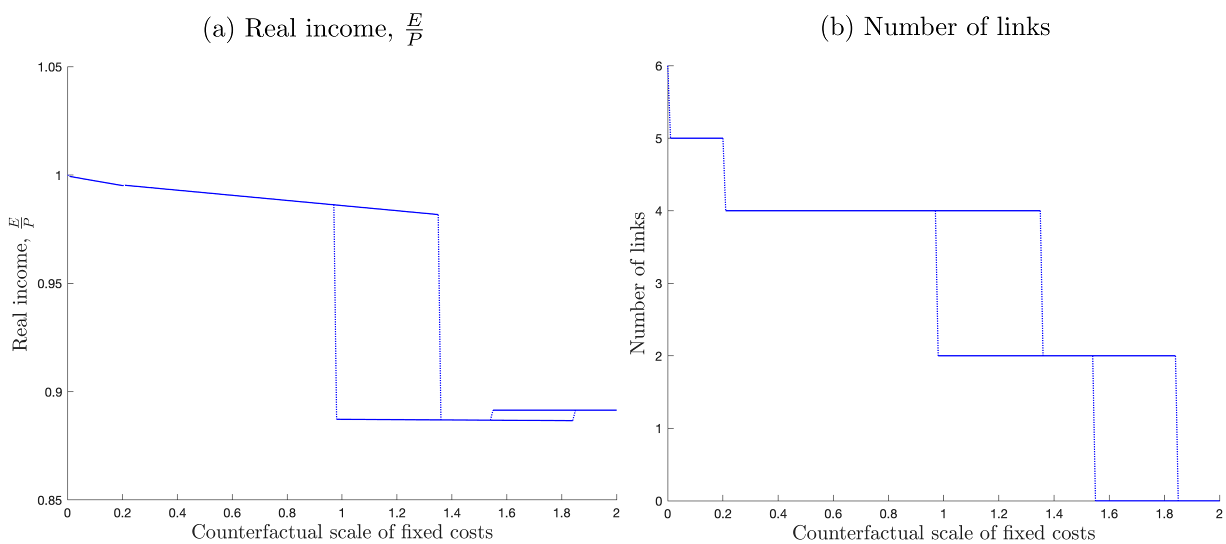
Notes: The figures report the responses of real income and number of links to counterfactual scales of fixed costs in an economy summarized in Figure 2. Fixed costs are scaled by factors indicated on the horizontal axis. Note that the baseline is indicated by scale equal to one. Real income is normalized with respect to the value when fixed costs are zero.

Figure A2: Counterfactual scale of fixed costs: endogenous cyclic network with buyers initiating contacts



Notes: The figures report the responses of real income and number of links to counterfactual scales of fixed costs in an economy summarized in Figure 3. Fixed costs are scaled by factors indicated on the horizontal axis. Note that the baseline is indicated by scale equal to one. Real income is normalized with respect to the value when fixed costs are zero.

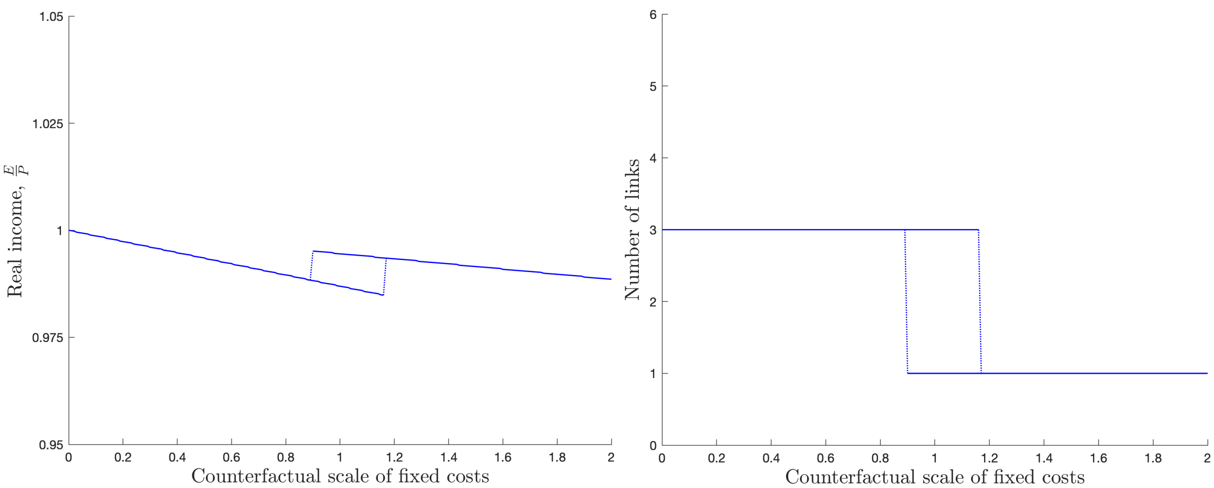
Figure A3: Counterfactual scale of fixed costs: endogenous cyclic network with suppliers initiating contacts



Notes: The figures report the responses of real income and number of links to counterfactual scales of fixed costs in an economy summarized in Figure 4. Fixed costs are scaled by factors indicated on the horizontal axis. Note that the baseline is indicated by scale equal to one. Real income is normalized with respect to the value when fixed costs are zero.

Figure A4: Counterfactual scale of fixed costs: endogenous acyclic network with suppliers initiating contacts

(a) Real income, $\frac{E}{P}$ (b) Number of links



Notes: The figures report the responses of real income and number of links to counterfactual scales of fixed costs in an economy summarized in Figure 5. Fixed costs are scaled by factors indicated on the horizontal axis. Note that the baseline is indicated by scale equal to one. Real income is normalized with respect to the value when fixed costs are zero.

A.5 Small Open Economy

In this section, we describe the model of a small open economy. As in the main text, we first describe the economy under fixed production networks and then extend the model to allow for endogenous network formation.

A.5.1 Fixed production networks

Preferences and Demand. Preferences and demand are characterized by equations (1), (2), and (3). We assume that domestic final consumers do not buy from foreign producers directly.

Market Structure and Production. The foreign supplier, denoted by F, is in each domestic firm's set of potential suppliers (i.e., $F \in \mathbf{Z}_j$). All domestic firms take the import prices $\{p_{Fj}\}_{j\in\Omega}$ as given. The marginal cost of firm j is characterized by equation (4).

Firm Cost Shares, Sales, and Profits. Equations (5) and (7) characterize the shares of variable costs by firm j that are spent on intermediate inputs produced by domestic firm $k \in Z_j$ and on labor, respectively. In addition, the import share of firm j (assuming $F \in Z_j$) is

$$s_{Fj} = \frac{p_F q_{Fj}}{c_j q_j} = \frac{\alpha_{Fj}^{\sigma-1} p_{Fj}^{1-\sigma}}{\Theta_j}. \tag{A3}$$

If firm j sells abroad, the demand from the foreign buyer takes a similar functional form:

$$q_{jF} = \beta_{jF}^{\sigma-1} \tau_{jF}^{-\sigma} p_{jF}^{-\sigma} D_F, \tag{A4}$$

where D_F is the foreign demand shifter that firms take as given, and τ_{jF} is the export costs that firms face when exporting.

Firms' total sales consist of the sum of sales to domestic final consumers, exports, and sales to other domestic firms. Let firm j's total sales be denoted by

$$\begin{aligned} x_j = \; & \underbrace{\beta_{jH}^{\sigma-1} \mu_{jH}^{1-\sigma} \phi_j^{\sigma-1} \Theta_j \frac{E}{P^{1-\sigma}}}_{\text{Sales to domestic final consumers}} + \underbrace{I_{jF} \beta_{jF}^{\sigma-1} \tau_{jF}^{1-\sigma} \mu_{jF}^{1-\sigma} \phi_j^{\sigma-1} \Theta_j D_F}_{\text{Exports}} \\ & + \underbrace{\sum_{k\in\Omega} \mathbb{I}\{j \in Z_k\} \alpha_{jk}^{\sigma-1} \mu_{jk}^{1-\sigma} \phi_j^{\sigma-1} \Theta_j \frac{x_k}{\Theta_k \bar{\mu}_k}}_{\text{Sales to other domestic firms}}, \end{aligned} \tag{A5}$$

where I_{jF} is a binary variable that indicates firm j's export participation decision. The variable profits of firm j are given by equation (10).

Aggregation and Equilibrium. As described in Section 2, we abstract from fixed costs of linkage formation in an economy with a fixed network: hence, $\pi_j = \pi_j^{var}$. The aggregate household expenditure is given by equation (11), and the labor market clearing condition is given by equation

(12). In addition, we impose a balanced trade condition that determines the domestic wage level, w:

$$\sum_{j\in\Omega} I_{jF}\beta_{jF}^{\sigma-1}\tau_{jF}^{1-\sigma}\mu_{jF}^{1-\sigma}\phi_j^{\sigma-1}\Theta_j D_F = \sum_{j\in\Omega}\frac{1}{\bar{\mu}_k}s_{Fj}x_j \tag{A6}$$

The equilibrium for the small open economy with a fixed network is defined as follows:

Definition 4. *Given*

a network structure, $\{Z_j\}_{j\in\Omega}$,
a set of markups for sales to domestic households and exports, $\{\mu_{jH}\}_{j\in\Omega}$ *and* $\{\mu_{jF}\}_{j\in\Omega}$,
a set of markups for sales to firms, $\{\mu_{jk}\}_{j,k\in\Omega}$,
an aggregate labor supply, L,
a foreign demand, D_F,
a set of export costs, $\{\tau_{jF}\}_{j\in\Omega}$, *and*
a set of prices by the foreign supplier, $\{p_{Fj}\}_{j\in\Omega}$,

a ***fixed network equilibrium for the small open economy*** *is characterized by*

a wage level, w,
firm-level costs, $\{c_j\}_{j\in\Omega}$,
firm-level labor demand, $\{\ell_j\}_{j\in\Omega}$,
quantities of goods purchased by domestic households, $\{q_{jH}\}_{j\in\Omega}$,
quantities of goods purchased by firms, $\{q_{jk}\}_{j,k\in\Omega}$,
quantities of imports and exports, $\{q_{Fj}\}_{j\in\Omega}$ *and* $\{q_{jF}\}_{j\in\Omega}$,
a price index for the consumer, P, *and*
an aggregate expenditure, E,

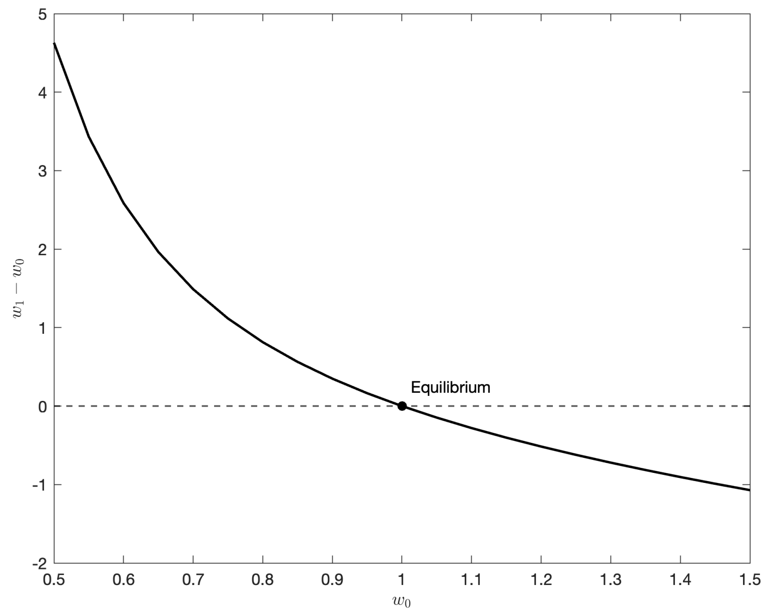
such that equations (2), (3), (4), (5), (7), (9), (10), (11), (12), (A3), (A4), (A5), *and* (A6) *hold.*

Existence and Uniqueness. Given the estimated parameters of the model, we are able to numerically examine the existence and uniqueness of the fixed network equilibrium. In a small open economy, the domestic wage, w, is now an additional equilibrium variable that is determined by the aggregate trade balance condition. One can follow the same steps as in Section 2.5 to establish the uniqueness of the other equilibrium variables, given a level of w. Therefore, it remains to characterize the uniqueness of w. Specifically, we consider different initial guesses w_0. We then compute the implied w_1 for each initial guess and plot the difference $w_1 - w_0$. A guess is an equilibrium when $w_1 - w_0 = 0$, and the equilibrium is unique when there is only one global minimum. Figure A5 shows that a unique fixed network equilibrium likely exists given the parameters listed in Table 3.

A.5.2 Endogenous network formation

We then extend our small open economy model to allow for endogenous network formation.

Figure A5: Difference between implied wage w_1 and initial guess w_0



Notes: The figure plots $w_1 - w_0$ as a function of w_0. Initial guesses w_0 are drawn from $[0.5, 1.5]$ discretized into 20 equally sized bins. We adopt the parameter values listed in Table 3.

Network Formation. We maintain the assumptions that buyers initiate link formations and that the network is acyclic. Given a sourcing strategy, Z_j, and export participation choice, I_{jF}, the profits of firm j are equal to variable profits less the fixed costs of domestic and foreign sourcing, $\sum_{k \in Z_j} f_{kj} w$, and the fixed costs of exporting, $I_{jF} f_{jF} w$:

$$\pi_j(Z_j, I_{jF}) = \pi_j^{var}(Z_j, I_{jF}) - \sum_{k \in Z_j} f_{kj} w - I_{jF} f_{jF} w. \tag{A7}$$

Firm j endogenously decides on the set of suppliers and whether or not to export given the set of potential suppliers, $\mathbf{Z}_j$:

$$\max_{Z_j, I_{jF}} \ \pi_j(Z_j, I_{jF}) \quad \text{s.t.} \quad Z_j \subseteq \mathbf{Z}_j, \, I_{jF} \in \{0, 1\}. \tag{A8}$$

Figure A6 shows the potential connections for domestic firms in the small open economy.

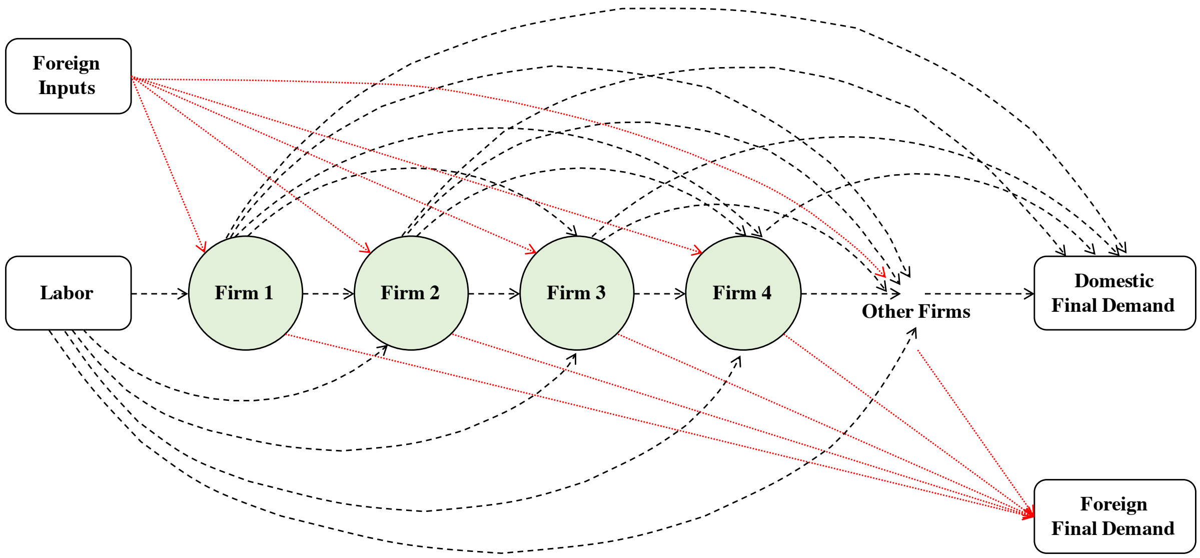
Aggregation and Equilibrium. Assume that there are no foreign asset holdings and that trade is balanced. The aggregate household expenditure is given by equation (11). Labor market clearing is given by equation (12). The balanced trade condition is given by equation (A6).

The equilibrium for the small open economy is defined as follows.

Definition 5. *Given*

a collection of potential supplier sets for each firm, $\{\mathbf{Z}_j\}_{j \in \Omega}$, that satisfies the acyclicity of the network,

Figure A6: Endogenous acyclic network formation: potential connections in a small open economy



common markups for sales to domestic households and exports, $\mu_{jH} = \mu_{jF} = \frac{\sigma}{\sigma-1}\ \forall j \in \Omega$*,*
common markups for sales to firms, $\mu_{jk} = 1\ \forall j, k \in \Omega$*,*
an aggregate labor supply, L*,*
a foreign demand, D_F*,*
a set of export costs, $\{\tau_{jF}\}_{j\in\Omega}$*, and*
a set of prices by the foreign supplier, $\{p_{Fj}\}_{j\in\Omega}$*,*

*an **endogenous network equilibrium with buyers initiating contacts for the small open economy** is characterized by*
a wage level, w*,*
firm-level costs, $\{c_j\}_{j\in\Omega}$*,*
firm-level labor demand, $\{\ell_j\}_{j\in\Omega}$*,*
quantities of goods purchased by domestic households, $\{q_{jH}\}_{j\in\Omega}$*,*
quantities of goods purchased by firms, $\{q_{jk}\}_{j,k\in\Omega}$*,*
quantities of imports and exports, $\{q_{Fj}\}_{j\in\Omega}$ *and* $\{q_{jF}\}_{j\in\Omega}$*,*
a price index for the consumer, P*,*
an aggregate expenditure, E*,*
a set of sourcing strategies, $\{Z_j\}_{j\in\Omega}$*, and*
export participation choices $\{I_{jF}\}_{j\in\Omega}$*,*

such that firms solve the optimization problem characterized by (A7) *and* (A8)*, and equations* (2)*,* (3)*,* (4)*,* (5)*,* (7)*,* (9)*,* (10)*,* (11)*,* (12)*,* (A3)*,* (A4)*,* (A5)*, and* (A6) *hold.*

Existence and Uniqueness. To ensure the existence of an equilibrium, we employ logit smoothing as described in Section A.3. Instead of making a binary decision on whether to export, firms choose the probability of exporting, denoted p_{jF}. Since the sourcing strategies depend on the exporting decision, firms solve the sourcing problem in two steps. First, conditional on export participation, I_{jF}, firm j chooses the probability of sourcing from each potential supplier, $\mathsf{p}_{kj}(I_{jF})$, as well as the probability of importing, $\mathsf{p}_{Fj}(I_{jF})$, from which the firm obtains the profits (value) given I_{jF}, denoted $v_j(I_{jF})$:

$$v_j(I_{jF}) = \left(1 - \frac{1}{\mu_{jH}}\right) \beta_{jH}^{\sigma-1} \mu_{jH}^{1-\sigma} \phi_j^{\sigma-1} \Theta_j(I_{jF}) A + I_{jF} \left(1 - \frac{1}{\mu_{jF}}\right) \beta_{jF}^{\sigma-1} \tau_{jF}^{1-\sigma} \mu_{jF}^{1-\sigma} \phi_j^{\sigma-1} \Theta_j(I_{jF}) D_F \\ - \sum_{k \in \mathbf{Z}_j} \mathsf{p}_{kj}(I_{jF}) f_{kj} w - I_{jF} f_{jF} w. \tag{A9}$$

Then, the firm chooses the probability of exporting given the value $v_j(I_{jF})$:

$$\mathsf{p}_{jF} = \frac{\exp(v_j(I_{jF} = 1)/\lambda)}{\exp(v_j(I_{jF} = 1)/\lambda) + \exp(v_j(I_{jF} = 0)/\lambda)}. \tag{A10}$$

The resulting sourcing probabilities are the average of two sets of conditional sourcing probabilities using exporting probability as a weight:

$$\mathsf{p}_{kj} = \mathsf{p}_{jF} \mathsf{p}_{kj}(I_{jF} = 1) + (1 - \mathsf{p}_{jF}) \mathsf{p}_{kj}(I_{jF} = 0). \tag{A11}$$

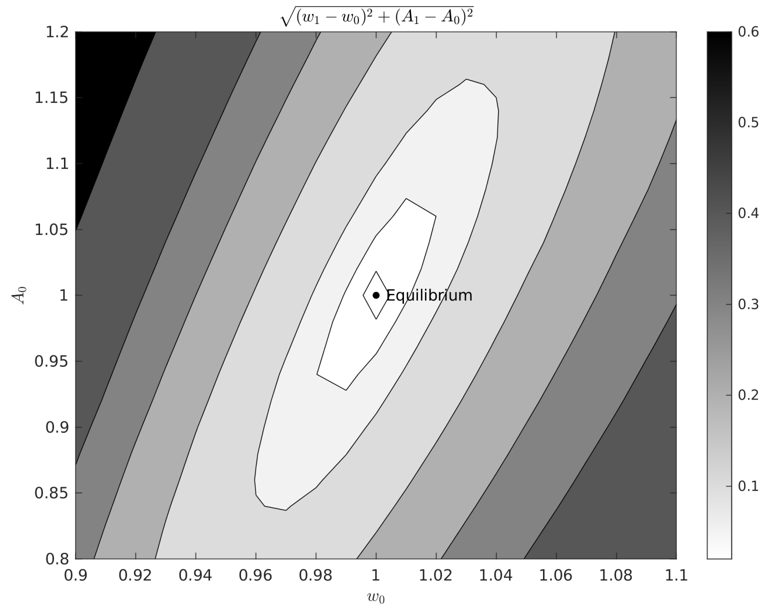
Given the estimated parameters of the model, we are able to numerically examine the existence and uniqueness of the equilibrium. As we need to search for two equilibrium variables instead of one, we consider different initial guesses (w_0, A_0) from discretized $[w_{min}, w_{max}] \times [A_{min}, A_{max}]$. We then compute the implied (w_1, A_1) for each initial guess and plot the Euclidean norm of the difference between (w_0, A_0) and (w_1, A_1). A guess is an equilibrium when it attains the smallest norm, and the equilibrium is unique when there is only one global minimum. Figure A7 shows that a unique equilibrium likely exists given the parameters listed in Table 3.

A.6 Solution Algorithm

A.6.1 Solution algorithm for endogenous network equilibrium

This section illustrates how firms' optimal sourcing problems can be solved sequentially, given an ordering of firms and a guess for domestic household demand, in an economy where the buyers initiate the link formation and all feasible production networks are acyclic. In the exogenously given ordering of firms, let firm 1 be the first in the sequence. This means that firm 1 can only hire labor inputs. To make its decision on how much labor to hire, firm 1 only needs to know the domestic household demand, A (wage is normalized to be one). Firm 2 is second in the sequence and can hire labor inputs as well as purchase the input produced by firm 1. To make its decision, firm 2 needs to know the domestic household demand level as well as the cost of its potential supplier (firm 1)—which is already determined after firm 1's decision. Firm 3 can hire labor and purchase the inputs from firm 1, firm 2, or both. To make its decision, firm 3 needs to know the two upstream firms' cost and the level of domestic household demand. After sequentially solving the problems of all firms, we obtain a set of sourcing strategies that is consistent with the initial

Figure A7: Norm of implied difference between (w_0, A_0) and (w_1, A_1)



Notes: The figure plots the Euclidean norm of the difference between (w_0, A_0) and (w_1, A_1). Initial guesses (w_0, A_0) are drawn from $[0.9, 1.1] \times [0.8, 1.2]$ discretized into 20×20 equally sized bins. We adopt the parameter values listed in Table 3.

guess for domestic household demand.

Each firm faces a discrete choice problem of selecting its own sourcing strategy given the decisions of upstream firms. In general, solving the problem is challenging when the number of potential suppliers is large. In our case, this challenge is partially resolved by the assumption that final demand and inputs have the same elasticity of substitution across products. Under this assumption, the scale and substitution effects offset each other, and the benefit and cost of adding another supplier are additively separable across suppliers. In other words, plugging equation (10) into equation (15) leads to a profit function that is linear in the elements of Z_j. Hence, firms can evaluate each potential supplier independently since the order of link formation does not matter.[27] These properties become useful when applying the iterative algorithm to solve the problem described in (15) for a given firm given its knowledge about the costs of the potential suppliers in $\mathbf{Z}_j$.

Although the assumptions above simplify the problem, we are still limited in the number of possible sourcing strategies we can feasibly evaluate. We therefore restrict the set of potential suppliers for firm j, $\mathbf{Z}_j$, to be a random subset from the set of firms prior to firm j in the sequence. The suppliers for firm j are then optimally chosen as the solution to the problem in (15). In practice, we choose the maximum cardinality of $\mathbf{Z}_j$ to be 1,000. The firm's order in the sequence of

[27]In an earlier version of the paper, we focused on the case with $\rho < \sigma$, where ρ is the elasticity of substitution in the production function. Under this case, there are complementarities in the firm's sourcing decisions (i.e., the firm's decision to source from a potential supplier affects its decision on whether to source from another supplier). We solved for the firms' optimal sourcing decisions using an iterative algorithm developed by Jia (2008). Our model in this paper demonstrates a simpler case in which there are no such complementarities when firms choose their sourcing strategies.

supplier choices and its set of potential suppliers are attributes of the firm and therefore primitives of the model.

Importantly, we are searching here only for a fixed point in one scalar (domestic household demand A) as opposed to a large vector of costs and searching strategies for all firms. In other words, the ordering approach implies that even with a rich micro structure and firm-level heterogeneity, knowing only one equilibrium variable is sufficient to solve the firms' problems sequentially.

Solution algorithm. The solution algorithm to compute the endogenous network equilibrium is as follows:

1. Guess a domestic household demand, A_0.
2. Given A_0, firms simultaneously solve for their sourcing strategies, $\{Z_j(A_0)\}_{j\in\Omega}$, and costs, $\{c_j(A_0)\}_{j\in\Omega}$. In practice, under the equilibrium in which buyers initiate contacts and networks are acyclic, one can order firms in a sequence and compute their sourcing strategies sequentially. Specifically,
   (a) Firm 1 can only hire labor inputs. Firm 1's sourcing strategy, Z_1, and cost, c_1, are determined.
   (b) Firm 2 can hire labor inputs as well as purchase the input produced by firm 1. Firm 2 decides whether to source from firm 1, $\mathbb{I}_{12}(A_0)$, by solving the problem defined in equation (15) based on c_1 and A_0. Firm 2's sourcing strategy, $Z_2(A_0)$, and cost, $c_2(A_0)$, are determined.
   (c) Firm j, $j > 2$, can hire labor inputs as well as purchase the input produced by eligible suppliers prior in the sequence ($k < j$). Firm j decides whether to source from an eligible supplier k, $\mathbb{I}_{kj}(A_0)$, by solving the problem defined in equation (15) based on $c_k(A_0)$ and A_0. Firm j's sourcing strategy, $Z_j(A_0)$, and cost, $c_j(A_0)$, are determined.
3. Given $\{Z_j(A_0)\}_{j\in\Omega}$ and $\{c_j(A_0)\}_{j\in\Omega}$, solve for the equilibrium as in the fixed network case. In particular, compute the implied domestic household demand, $A_1(A_0) = \frac{E(A_0)}{P(A_0)^{1-\sigma}}$. The implied price index, $P(A_0)$, is computed from equations (3) and (4). The implied aggregate income, $E(A_0)$, is computed from equation (13).
4. Update initial guess A_0 given $A_1(A_0)$. Repeat the steps above until A converges.

Solution algorithm with logit smoothing. The solution algorithm to compute the endogenous network equilibrium with logit smoothing is as follows:

1. Guess a domestic household demand, A_0.
2. Given A_0, firms simultaneously solve for their sourcing strategies, $\{Z_j(A_0)\}_{j\in\Omega}$, and costs, $\{c_j(A_0)\}_{j\in\Omega}$. In practice, under the equilibrium in which buyers initiate contacts and networks are acyclic, one can order firms in a sequence and compute their sourcing strategies sequentially. Specifically,

(a) Firm 1 can only hire labor inputs. Firm 1's sourcing strategy, Z_1, and cost, c_1, are determined.

(b) Firm 2 can hire labor inputs as well as purchase the input produced by firm 1. Firm 2 decides the probability of sourcing from firm 1, $\mathsf{p}_{12}(A_0)$, based on c_1 and A_0 using the logit smoothing technique described in Appendix A.3. Firm 2's sourcing strategy, $Z_2(A_0)$, and cost, $c_2(A_0)$, are determined.

(c) Firm j, $j > 2$, can hire labor inputs as well as purchase the input produced by eligible suppliers prior in the sequence ($k < j$). Firm j decides the probability of sourcing from an eligible supplier k, $\mathsf{p}_{kj}(A_0)$, based on $c_k(A_0)$ and A_0 using the logit smoothing technique described in Appendix A.3. Firm j's sourcing strategy, $Z_j(A_0)$, and cost, $c_j(A_0)$, are determined.

3. Given $\{Z_j(A_0)\}_{j\in\Omega}$ and $\{c_j(A_0)\}_{j\in\Omega}$, solve for the equilibrium as in the fixed network case. In particular, compute the implied domestic household demand, $A_1(A_0) = \frac{E(A_0)}{P(A_0)^{1-\sigma}}$. The implied price index, $P(A_0)$, is computed from equations (3) and (4). The implied aggregate income, $E(A_0)$, is computed from equation (13).

4. Update initial guess A_0 given $A_1(A_0)$. Repeat the steps above until A converges.

Solution algorithm for a small open economy with logit smoothing. The equilibrium of a small open economy is characterized by both domestic household demand, A, and domestic wage, w. Therefore, one needs to find the fixed point of (A, w). To apply the algorithm (with logit smoothing) above, in step 1, guess a domestic household demand, A_0, and a domestic wage, w_0. In step 2, solve for the set of exporting probabilities of all firms, $\{\mathsf{p}_{jF}(A_0, w_0)\}_{j\in\Omega}$, in addition to $\{Z_j(A_0, w_0), c_j(A_0, w_0)\}_{j\in\Omega}$. Note that one needs to take into account the interdependence between exporting and sourcing decisions for firms other than firm 1. Specifically, for firm $j > 1$ in step 2.(b) or 2.(c),

(i) For each possible export participation decision, $I_{jF} \in \{0, 1\}$, determine firm j's (conditional) probability of sourcing from each eligible supplier k, $\mathsf{p}_{kj}(I_{jF}, A_0, w_0)$, given (A_0, w_0) and costs of eligible suppliers using the logit smoothing technique described in Appendix A.3. Firm j's (conditional) sourcing strategy, $Z_j(I_{jF}, A_0, w_0)$, and (conditional) cost, $c_j(I_{jF}, A_0, w_0)$, are determined.

(ii) Compute firm j's profits for each possible export participation decision from equation (A9) in Appendix A.5 and calculate its probability of exporting, $\mathsf{p}_{jF}(A_0, w_0)$, from equation (A10) in Appendix A.5.

(iii) Compute firm j's (unconditional) probability of sourcing from each eligible supplier k, $\mathsf{p}_{kj}(A_0, w_0)$, from equation (A11) in Appendix A.5. Firm j's (unconditional) sourcing strategy, $Z_j(A_0, w_0)$, and (unconditional) cost, $c_j(A_0, w_0)$, are determined.

In step 3, solve for the implied domestic household demand, $A_1(A_0, w_0)$, as well as implied domestic wage, $w_1(A_0, w_0)$, given $\{\mathsf{p}_{jF}(A_0, w_0), Z_j(A_0, w_0), c_j(A_0, w_0)\}_{j\in\Omega}$. In step 4, update initial guess of (A_0, w_0) and repeat the steps above until (A, w) converges.

A.6.2 Hat algebra for fixed network equilibrium

When the network structure is fixed, the counterfactual equilibrium can be solved using hat algebra (Dekle, Eaton, and Kortum, 2007). Given exogenous changes in export costs, $\hat{\tau}_{jF}$, and import prices, $\hat{p}_{Fj}$, the fixed network equilibrium can be solved by searching for the change in wage $\hat{w}$ that clears all markets. We adopt the following procedures:

1. Guess change in wage, $\hat{w}$.
2. Compute change in $\hat{c}_j^{1-\sigma}$:

$$\hat{c}_j^{1-\sigma} = s_{Lj}^{Total}\hat{w}^{1-\sigma} + \hat{t}_{Fj}.$$

   The total labor share s_{Lj}^{Total} is defined as $1 - s_{Fj}^{Total}$, where the total import share s_{Fj}^{Total} is obtained from solving $s_{Fj}^{Total} = s_{Fj} + \sum_{k\in\mathbf{Z}_j\setminus\{F\}} s_{kj}s_{Fk}^{Total}$. The term $\hat{t}_{Fj}$ is obtained by solving the following:

$$\hat{t}_{Fj} = s_{Fj}\hat{p}_{Fj}^{1-\sigma} + \sum_{k\in\mathbf{Z}_j} s_{kj}\hat{t}_{Fk}.$$

3. Compute change in exports, $\hat{x}_{jF}$:

$$\hat{x}_{jF} = \hat{c}_j^{1-\sigma}\hat{\tau}_{jF}^{1-\sigma}.$$

4. Compute change in domestic household demand, $\hat{A}$:

$$\hat{A} = \frac{\hat{E}}{\hat{P}^{1-\sigma}}, \quad \hat{E} = \frac{1}{E}\left(\mu w L^{var}\hat{w} + (\mu - 1)\sum_{j\in\Omega} x_{jF}\hat{x}_{jF}\right), \quad \hat{P}^{1-\sigma} = \sum_{j\in\Omega} s_{jH}\hat{c}_j^{1-\sigma},$$

   where $L^{var} = L - \sum_{j\in\Omega}\left(\sum_{k\in\mathbf{Z}_j} \mathsf{p}_{kj}f_{kj} - \mathsf{p}_{jF}f_{jF}\right)$ is the total variable labor supply. Note that we obtain the above equation for $\hat{E}$ from the identity:

$$E = wL^{var} + \frac{\mu - 1}{\mu}E + \frac{\mu - 1}{\mu}\sum_{j\in\Omega} x_{jF},$$

5. Compute the change in sales to domestic household demand, $\hat{x}_{jH}$:

$$\hat{x}_{jH} = \hat{c}_j^{1-\sigma}\hat{A}.$$

6. Compute changes in direct foreign input share, $\hat{s}_{Fj}$, and firm-to-firm share, $\hat{s}_{kj}$:

$$\hat{s}_{Fj} = \frac{\hat{p}_{Fj}^{1-\sigma}}{\hat{c}_j^{1-\sigma}}, \quad \hat{s}_{kj} = \frac{\hat{c}_k^{1-\sigma}}{\hat{c}_j^{1-\sigma}}.$$

   Moreover, solve for the total share of foreign input in the counterfactual equilibrium using $s_{Fj}^{Total'} = s'_{Fj} + \sum_{k \in \mathbf{Z}_j} s'_{kj} s_{Fk}^{Total'}$ and compute the corresponding total labor share, $s_{Lj}^{Total'} = 1 - s_{Fj}^{Total'}$.

7. Compute the implied change in wage, $\hat{w}'$:

$$\hat{w}' = \frac{1}{wL^{var}} \frac{1}{\mu} \sum_{j \in \Omega} s_{Lj}^{Total'} (x_{jH} \hat{x}_{jH} + x_{jF} \hat{x}_{jF}).$$

   The above is obtained from the identity:

$$wL^{var} = \frac{1}{\mu} \sum_{j \in \Omega} s_{Lj}^{Total} (x_{jH} + x_{jF}).$$

8. Update $\hat{w}$ by taking the weighted average of the initial guess $\hat{w}$ and the implied wage change $\hat{w}'$:

$$\hat{w} = (1 - \lambda)\hat{w} + \lambda \hat{w}', \quad \lambda \in (0, 1).$$

9. Repeat the steps above until $\hat{w}$ converges.

B Ordering Algorithm

In this section, we describe the implementation of the ordering algorithm to solve the feedback arc set problem. We begin by defining some terms and notation.

B.1 Terms and Notation

- *graph / network*, $G = (V, E)$ - A collection of a set of edges E and set of vertices V. Edges describe the relationship between vertices. Two basic classifications of graphs are based on whether the edges are *directed* or *undirected* and whether they are *weighted* or *unweighted*
- $n = |V|$, $m = |E|$
- *cycle* - A path within a graph where a vertex is reachable from itself
- $d^+(u)$ - For a vertex $u \in V$ in a directed graph, number of outgoing edges
- $d^-(u)$ - For a vertex $u \in V$ in a directed graph, number of incoming edges

- $w^+(u)$ - For a vertex $u \in V$ in a directed graph, cumulative sum of weights of outgoing edges
- $w^-(u)$ - For a vertex $u \in V$ in a directed graph, cumulative sum of weights of incoming edges
- *sink* - A vertex $u \in V$ in a directed graph with $d^+(u) = 0$
- *source* - A vertex $u \in V$ in a directed graph with $d^-(u) = 0$
- *feedback arc set* - A set of edges from a directed cyclic graph that when removed make the graph acyclic
- $s = s_{left}s_{right}$ - Given two finite sequences s_{left} and s_{right} with the indicated notation, we symbolize the *concatenation* operation. For example, if $s_{left} = (A, B, C)$ and $s_{right} = (X, Y, Z)$, then $s = s_{left}s_{right} = (A, B, C, X, Y, Z)$
- $\lfloor x \rfloor$ is the greatest integer less than or equal to x

B.2 Overview

The Belgian B2B data describe a weighted directed graph $G = (V, E)$. Vertices are firms, and edges are sales between firms. The goal of the ordering algorithm is to order firms in a way such that a given firm only sells to firms further along in the ordering and only buys from firms that precede it. The condition desired by this ordering is known in graph theory as a *topological ordering* (Black, 1999). A topological ordering exists if and only if a graph is directed and acyclic. The B2B data are cyclic. For the *unweighted* case, our motivation is to find a feedback arc set of minimal cardinality: that is, what is the minimum number of transactions that we need to drop (i.e., the "violators") from our network to satisfy our ordering condition? For the *weighted case*, we seek to find a feedback arc set such that the cumulative weight of the violating transactions is minimized. Finding a minimum feedback arc set is computationally difficult, but approximation algorithms exist.

B.3 Unweighted Case

The algorithm we use for the paper was first presented by Eades et al. (1993). This algorithm was chosen because of its linear run time complexity, $O(m+n)$, and because of its relative simplicity in implementation. The algorithm uses a greedy heuristic through which it builds the proposed ordering $s = s_{left}s_{right}$.[28] Vertices are initialized into several buckets: sinks, sources, and δ buckets, where for a vertex $u \in V$, $\delta(u) = d^-(u) - d^+(u)$.[29] At each iteration, the algorithm removes all sinks from the network and prepends them to a sequence s_{right}, removes all sources and appends

[28]According to Black (2005), a greedy algorithm is "an algorithm that always takes the best immediate, or local, solution while finding an answer. Greedy algorithms find the overall, or globally, optimal solution for some optimization problems, but may find less-than-optimal solutions for some instances of other problems."

[29]We have flipped the sign here compared to Eades et al. (1993) to be consistent with the diagrams elsewhere in our paper.

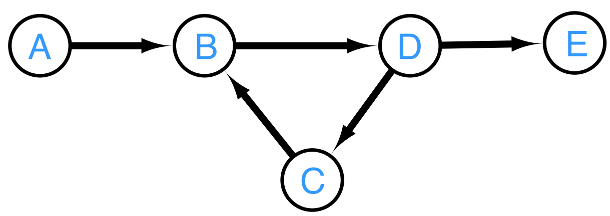
them to a sequence s_{left}, and then removes the vertex with the lowest δ score (the most "source"-like vertex) and appends it to s_{left}.[30] Each removal requires updating the buckets to reflect the modified graph. The algorithm stops when the graph is empty. There will be $2n-1$ buckets, which can be formalized as follows:[31]

$$V_{-n+1} = V_{sources} = \{u \in V \mid d^-(u) = 0;\ d^+(u) > 0\}$$
$$V_{n-1} = V_{sinks} = \{u \in V \mid d^-(u) > 0;\ d^+(u) = 0\}$$
$$V_d = \{u \in V \mid d = \delta(u);\ d^-(u) > 0;\ d^+(u) > 0\}$$

The bucket V_{-n+1} contains all the vertices that are only the sources of edges. The bucket V_{n-1} contains all the vertices that are only the sinks of edges (in other words, vertices that are only receiving edges). Each V_d bucket contains vertices with d net incoming edges (conditional on these vertices having both outgoing and incoming edges).

B.4 Example Execution on Unweighted Network

Consider the following network:



Let's trace the execution of the algorithm described by Eades et al. (1993).

B.4.1 Initialization

Buckets:

A			D	C	B			E
sources	-3	-2	-1	0	1	2	3	*sinks*

$Ordering:\ s = s_{left} = s_{right} = ()$

[30]Eades et al. (1993) take the vertex with the maximum δ score.
[31]Eades et al. (1993) assume that the graph G is simple (no bidirectional edges), and hence their original algorithm only requires $2n-3$ buckets.

B.4.2 First iteration:

Remove sinks

Updated buckets:

A				C, D	B			
sources	-3	-2	-1	0	1	2	3	*sinks*

Updated ordering : $s_{left} = ()$, $s_{right} = (E)$, $s = s_{left}s_{right} = (E)$

Remove sources

Updated buckets:

				C, D, B				
sources	-3	-2	-1	0	1	2	3	*sinks*

Updated ordering : $s_{left} = (A)$, $s_{right} = (E)$, $s = s_{left}s_{right} = (A, E)$

Remove vertex with lowest delta score

Updated buckets:

B								D
sources	-3	-2	-1	0	1	2	3	*sinks*

Updated ordering : $s_{left} = (A, C)$, $s_{right} = (E)$, $s = s_{left}s_{right} = (A, C, E)$

B.4.3 Second iteration

Remove sinks

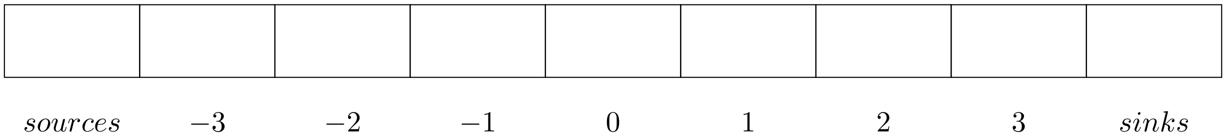
Updated buckets:

B								
sources	-3	-2	-1	0	1	2	3	*sinks*

Updated ordering : $s_{left} = (A, C)$, $s_{right} = (D, E)$, $s = s_{left}s_{right} = (A, C, D, E)$

Remove sources

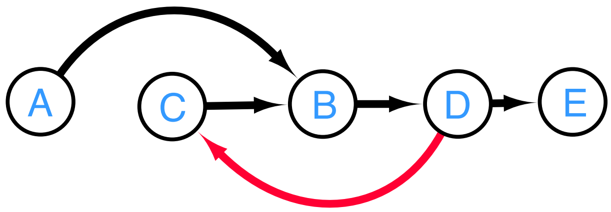
Updated buckets:



Updated ordering : $s_{left} = (A, C, B)$, $s_{right} = (D, E)$, $s = s_{left}s_{right} = (A, C, B, D, E)$

B.4.4 Final output

Ordering : $s = s_{left}s_{right} = (A, C, B, D, E)$, *Violator edge set*: $\{(D, C)\}$



B.5 Weighted Case

Simpson et al. (2016) have proposed a modification to adapt the Eades et al. (1993) algorithm to solve the weighted problem. The required changes are:

1. In the initialization step, all edge weights need to be normalized to be between 0 and 1.

2. $\delta(u)$ is redefined as $\delta(u) = \lfloor w^-(u) - w^+(u) \rfloor$.

The key motivation behind these steps is to reformat the network so that the unweighted version of the algorithm could be used in an identical fashion as before, specifically without increasing the number of buckets. Without the floor in step 2, for any given network the number of buckets could become large.

C Sensitivity Results

C.1 Real Income Changes under Cyclic and Acyclic Networks

Table C1 reports how the change in real income is affected when one considers an acyclic network structure. In the first and third columns, we take the network structure of year 2012 and compute the real income change in response to the changes in the export cost. In the second and fourth columns, we make use of the acyclic network from the algorithm explained in Appendix B for the weighted case where we minimize the values of violating transactions.

Table C1: Change in real income under cyclic and acyclic networks

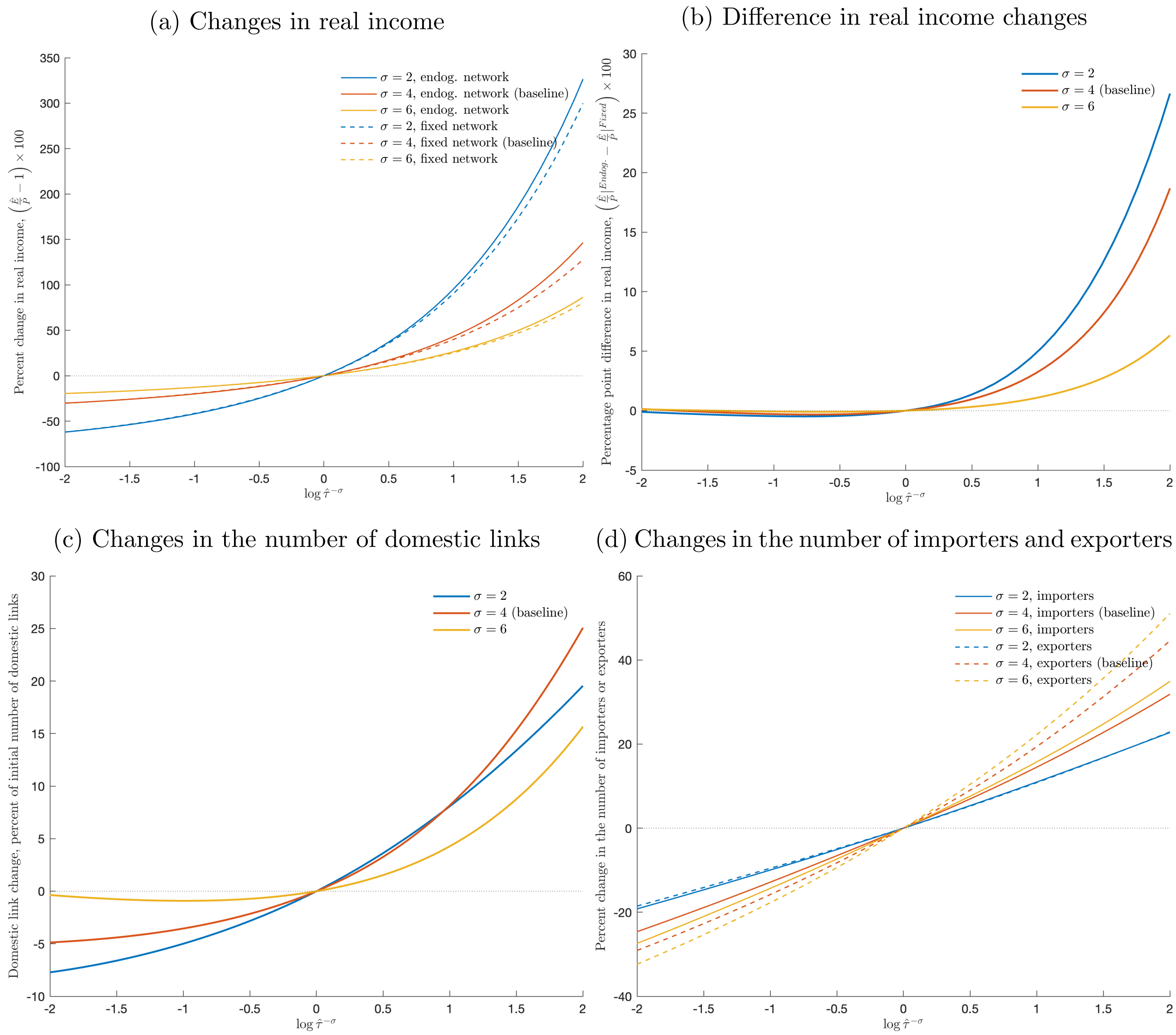
20% increase in $\tau_{\cdot F}$		Autarky	
$\frac{\hat{E}}{\hat{P}}$, cyclic	$\frac{\hat{E}}{\hat{P}}$, acyclic	$\frac{\hat{E}}{\hat{P}}$, cyclic	$\frac{\hat{E}}{\hat{P}}$, acyclic
0.7904	0.7831	0.5733	0.5456

Notes: For computing the changes in real income, we assume the value of σ to be 4.

C.2 Real Income Changes under Different Elasticities of Substitution

Figure C8 reports the aggregate responses to changes in export costs scaled by $-\sigma$ for $\sigma \in \{2, 4, 6\}$. Note that given σ, a change in export costs, $\hat{\tau}_{jF} = \hat{\tau}\ \forall j \in \Omega$, is equivalent to changing the foreign demand shifter, D_F, by $\hat{\tau}^{-\sigma}$ in terms of real income effect. Therefore, Figure C8 equivalently reports the aggregate responses to changes in the foreign demand shifter. In general, varying σ may generate non-monotonic real income effects of endogenous networks because σ affects the underlying parameters of the economy in complex ways, complicating the real income responses to trade shocks. Nevertheless, we find that the endogeneity of the extensive margin of firm-to-firm linkages tends to be more important the lower the elasticity of substitution between firms in the production and utility functions.

Figure C8: Aggregate responses to changes in export costs under different elasticities of substitution



Notes: The figure reports the aggregate responses to changes in the costs of exporting scaled by $-\sigma$ (with $\log \hat{\tau}^{-\sigma}$ on the horizontal axis) under different elasticities of substitution. For each value of $\sigma \in \{2, 4, 6\}$, we calibrate the parameters governing the distribution of productivity and estimate remaining model parameters as described in Section 4.3. We compute counterfactual relative real income changes for different export cost changes as described in Section 4.4. For each σ and $\hat{\tau}$, we report the mean estimates across 10 simulation draws.